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Eisenstein Series and Applications

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To Robert Langlands, on the occasion of his seventieth birthday

Preface

The theory of Eisenstein series, in the general form given to it by Robert Langlands some forty years ago, has been an important and incredibly useful tool in the fields of automorphic forms, representation theory, number theory and arithmetic geometry. For example, the theory of automorphic L-functions arises out of the calculation of the constant terms of Eisenstein series along parabolic subgroups. Not surprisingly, the two primary approaches to the analytic properties of automorphic L-functions, namely the Langlands–Shahidi method and the Rankin–Selberg method, both rely on the theory of Eisenstein series. In representation theory, Eisenstein series were originally studied by Langlands in order to give the spectral decomposition of the space of L^2-functions of locally symmetric spaces attached to adelic groups. This spectral theory has been used to prove the unitarity of certain local representations. Finally, on the more arithmetic side, the Fourier coefficients of Eisenstein series contain a wealth of arithmetic information which is far from being completely understood. The p-divisibility properties of these coefficients, for example, are instrumental in the construction of p-adic L-functions.

In short, the theory of Eisenstein series seems to have, hidden within it, an inexhaustible number of treasures waiting to be discovered and mined.

With such diverse applications, it is not easy even for the conscientious researcher to keep abreast of current developments. Indeed, different users of Eisenstein series often focus on different aspects of the theory. With this in mind, the workshop "Eisenstein Series and Applications" was held at the American Institute of Mathematics (Palo Alto) from August 15 to 19, 2005. The goal of the workshop was to bring together users of Eisenstein series from different areas who do not normally interact with each other, with the hope that such a juxtaposition of perspectives would provide deeper insight into the arithmetic of Eisenstein series and foster fruitful new collaborations.

This volume contains a collection of articles related to the theme of the workshop. Some, but not all of them, are based on lectures given in the workshop. We hope that the articles assembled here will be useful to a diverse audience and especially to students who are just entering the field.

We would like to take this opportunity to thank all the participants of the workshop for their enthusiastic participation, and the authors who contributed articles to this volume for their efforts and timely submissions, as well as, all the referees who gave the articles their thoughtful considerations. We are grateful to the American Institute of Mathematics and the National Science Foundation for providing generous support, and especially to Brian Conrey, David Farmer and Helen Moore of AIM for their invaluable assistance in the organization of the workshop.

We find it appropriate to dedicate this volume to Robert Langlands, who started it all, on the occasion of his seventieth birthday.

New York
August 2007 *Wee Teck Gan, Stephen Kudla and Yuri Tschinkel*

Contents

Twisted Weyl Group Multiple Dirichlet Series: The Stable Case

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Summary. *Weyl group multiple Dirichlet series* were associated with a root system Φ and a number field F containing the n-th roots of unity by Brubaker, Bump, Chinta, Friedberg, and Hoffstein [2]. Brubaker, Bump, and Friedberg [4] provided for when n is sufficiently large; the coefficients involve n-th order Gauss sums and reflect the combinatorics of the root system. Conjecturally, these functions coincide with Whittaker coefficients of metaplectic Eisenstein series, but they are studied in these papers by a method that is independent of this fact. The assumption that n is large is called *stability* and allows a simple description of the Dirichlet series. "Twisted" Dirichet series were introduced in Brubaker, Bump, Friedberg, and Hoffstein [5] without the stability assumption, but only for root systems of type A_r. Their description is given differently, in terms of Gauss sums associated to Gelfand–Tsetlin patterns. In this paper, we reimpose the stability assumption and study the twisted multiple Dirichlet series for general Φ by introducing a description of the coefficients in terms of the root system similar to that given in the untwisted case in [4]. We prove the analytic continuation and functional equation of these series, and when $\Phi = A_r$ we also relate the two different descriptions of multiple Dirichlet series given here and in [5] for the stable case.

1 Introduction

The present paper continues the study of Weyl group multiple Dirichlet series: families of Dirichlet series in several complex variables associated to a reduced root system Φ, a positive integer n, and a number field F containing the n-th roots of unity. These series take the form

$$Z_\Psi(\mathbf{s}, \mathbf{m}) = \sum_{\mathbf{c}} H\Psi(\mathbf{c}; \mathbf{m}) \mathbb{N}(\mathbf{c})^{-2\mathbf{s}} \tag{1}$$

where the ingredients in the above formula are defined as follows. Letting r denote the rank of the root system Φ, $\mathbf{s} = (s_1, \ldots, s_r) \in \mathbb{C}^r$, $\mathbf{m} = (m_1, \ldots, m_r)$

is a fixed r-tuple of nonzero $\mathfrak{o}_S$ integers, and the sum runs over $\mathbf{c} = (\mathbf{c}_1, \ldots, \mathbf{c_r})$, r-tuples of nonzero integral $\mathfrak{o}_S$-ideals. Here $\mathfrak{o}_S$ is the ring of S-integers of F, for a sufficiently large set of primes S containing the ramified places such that $\mathfrak{o}_S$ is a PID.

The coefficients of this multiple Dirichlet series involve the two functions Ψ and H. The function H carries the main arithmetic information. It is defined in terms of Gauss sums formed with n-th power residue symbols and a cocycle multiplicativity. An exact description is given in Section 4. The weight function Ψ lies in a finite dimensional space of locally constant functions on $\mathbb{A}_{F,f}^{\times}$, the finite idèles of F, and is described precisely in Section 2.3. Its presence guarantees that $H\Psi$ is well-defined up to $\mathfrak{o}_S$-units.

In what follows, we prove that $Z_\Psi(\mathbf{s}, \mathbf{m})$ has an analytic continuation to a meromorphic function on $\mathbb{C}^r$ and possesses a group of functional equations isomorphic to the Weyl group of Φ. See Theorem 14 for a precise statement. In the last section of the paper, we further explain how the series $Z_\Psi(\mathbf{s}, \mathbf{m})$, which generalize earlier results of [4], agree with the Weyl group multiple Dirichlet series defined in [5] when both series can be defined according to the data Φ and n. In the remainder of this introduction, we explain how these Weyl group multiple Dirichlet series are connected to Fourier–Whittaker coefficients of Eisenstein series, and further orient the reader as to how this paper fits within the framework of earlier works on the subject.

1.1 Connections to Eisenstein series

Fourier–Whittaker coefficients of Eisenstein series on reductive algebraic groups G contain Dirichlet series in several complex variables of arithmetic interest. Metaplectic groups — certain central extensions of the adèle points of split, reductive groups G by n-th roots of unity — have Whittaker coefficients that contain Dirichlet series that are "twisted" by n-th order characters. For example, nonvanishing of twists of GL(2) automorphic forms by quadratic or cubic characters may be proved in this way. (See Bump, Friedberg, and Hoffstein [8] and Brubaker, Friedberg, and Hoffstein [6].) Unfortunately, computing these Whittaker coefficients on higher rank metaplectic groups yields intractable exponential sums. So even though the resulting Dirichlet series inherits a Weyl group of functional equations, it is extremely difficult to directly realize it as explicitly consisting of recognizable arithmetic functions.

Motivated by the theory of metaplectic Eisenstein series, one may attempt to construct Dirichlet series in several complex variables with similar properties. In [2] and [4], a family of Weyl group multiple Dirichlet series are described using data consisting of a fixed positive integer n and a number field F containing the group μ_n of n-th roots of unity, together with a reduced root system Φ. The group of functional equations of these multiple Dirichlet series is similarly isomorphic to the Weyl group W of Φ.

Conjecturally, the Weyl group multiple Dirichlet series are Whittaker coefficients of metaplectic Eisenstein series. To be precise, let G be a split, simply-connected, semisimple algebraic group whose root system is the dual root system $\hat{\Phi}$, and let $\tilde{G}(\mathbb{A})$ be the n-fold metaplectic cover of $G(\mathbb{A})$ — where $\mathbb{A}$ is the adèle ring of F — as constructed by Kubota [11] and Matsumoto [13]. Let U be the unipotent radical of the standard Borel subgroup of G. The metaplectic cover splits over U, and we identify $U(\mathbb{A})$ with its image in $\tilde{G}(\mathbb{A})$. If α is a root of Φ, let $i_\alpha : \mathrm{SL}_2 \longrightarrow G$ be the embedding corresponding to a Chevalley basis of Lie(G). We consider the additive character $\psi_U : U(\mathbb{A})/U(F) \longrightarrow \mathbb{C}$ such that for each simple positive root α, the composite $\psi_U \circ i_\alpha \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}$ is a fixed additive character ψ of $\mathbb{A}/F$. Here $\mathfrak{o}_v$ is assumed the conductor of ψ for any prime $v \notin S$, where S is a finite set of places as described above.

Now let $E(g, s_1, \ldots, s_r)$ be an Eisenstein series of Borel type on $\tilde{G}(\mathbb{A})$. The coefficient

$$Z(s_1, \ldots, s_r) = \int_{U(\mathbb{A})/U(F)} E(u, s_1, \ldots, s_r)\, \psi_U(u)\, du$$

is a multiple Dirichlet series whose group of functional equations is isomorphic to W. We conjecture this to be the same as the multiple Dirichlet series described in [2] and [4].

By contrast, the Weyl group multiple Dirichlet series in this paper have an additional parameter $\mathbf{m} = (m_1, \ldots, m_r)$, an r-tuple of nonzero $\mathfrak{o}_S$ integers. To any such r-tuple, let $\psi_{U,\mathbf{m}} : U(\mathbb{A})/U(F) \longrightarrow \mathbb{C}$ be such that $\psi_{U,\mathbf{m}}\, i_{\alpha_i} \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}$ has conductor $p^{\mathrm{ord}_p(m_i)}$ for each prime p of $\mathfrak{o}_S$. Then conjecturally, the Weyl group multiple Dirichlet series are:

$$Z_\Psi(\mathbf{s}, \mathbf{m}) = Z(s_1, \ldots, s_r; m_1, \ldots, m_r) = \int_{U(\mathbb{A})/U(F)} E(u, s_1, \ldots, s_r)\, \psi_{U,m}(u)\, du.$$

Here E depends on Ψ. An alternate definition of Weyl group multiple Dirichlet series was given in [5], for which this equality is proved for $G = \mathrm{GL}_3$. Our verification that the series $Z_\Psi(\mathbf{s}, \mathbf{m})$ of this paper and the series of [5] agree in certain cases thus proves this equality for $Z_\Psi(\mathbf{s}, \mathbf{m})$ for these cases.

Throughout the remainder of this paper, the theory of Eisenstein series is suppressed, except for rank one Eisenstein series that underlie the proofs. As in [4], we use combinatorics and the geometry of the Weyl group to reduce to the case of Eisenstein series on the n-fold metaplectic cover of SL_2, whose functional equations were originally proved by Kubota [12], following the methods of Selberg and Langlands. While the definition of the coefficients of the Weyl group multiple Dirichlet series, given precisely in Section 4, may appear rather ad-hoc, it was arrived at by considerations connected with Eisenstein series, including early versions of computations that are included in [5], where the Whittaker coefficients of metaplectic Eisenstein series on GL_3 were worked out. Again, a direct approach to these Dirichlet series based on Eisenstein series leads to combinatorial complications that we are able to avoid by the present approach.

1.2 Connections to earlier results on multiple Dirichlet series

We conclude this introduction by explaining more precisely how this paper extends the results of earlier work in [4] and connects this approach to that of [5]. As noted above, Weyl group multiple Dirichlet series are defined in [4] for any reduced root system Φ and shown to possess a Weyl group of functional equations. However, the setting in [4] is specialized in two ways.

- We require that the integer n is "large enough," depending on Φ. We refer to this condition as the "stability assumption."
- The Dirichlet series is "untwisted" in a sense that will be made precise below upon comparison with other examples.

With these assumptions, the Dirichlet series admits a simple description. Using similar notation to (1) above, we may denote the coefficients of the Dirichlet series in [4] by $H(\mathbf{c}) = H(C_1, \ldots, C_r)$, where the C_i are now nonzero elements (rather than ideals) of the ring $\mathfrak{o}_S$ of S-integers, since we have chosen S so that $\mathfrak{o}_S$ is a principal ideal domain.

- The coefficients exhibit a twisted multiplicativity. This means that the Dirichlet series is not an Euler product, but specification of the coefficients is reduced to the specification of $H(p^{k_1}, \ldots, p^{k_r})$, where p is a fixed prime of $\mathfrak{o}_S$.
- Given $(k_1, \ldots, k_r)$, the coefficient $H(p^{k_1}, \ldots, p^{k_r})$ is zero unless there exists a Weyl group element $w \in W$ such that $\rho - w(\rho) = \sum k_i \alpha_i$, where ρ is half the sum of the positive roots in Φ, and $\alpha_1, \ldots, \alpha_r$ are the simple positive roots. If this is true, then $H(p^{k_1}, \ldots, p^{k_r})$ is a product of $l(w)$ n-th order Gauss sums, where $l : W \longrightarrow \mathbb{Z}$ is the length function.

We call the associated coefficients of the multiple Dirichlet series "untwisted, stable" coefficients owing to the special restrictions above.

In [5], "twisted" Weyl group multiple Dirichlet series are studied using a rather different perspective. The twisted Dirichlet series involve coefficients that we will denote $H_{GT}(\mathbf{c}; \mathbf{m}) = H_{GT}(C_1, \ldots, C_r; m_1, \ldots, m_r)$, to distinguish them from the coefficients $H(\mathbf{c}, \mathbf{m})$ of this paper. Roughly, these are twists of the coefficients $H_{GT}(C_1, \ldots, C_r; 1, \ldots, 1)$ described above by a set of n-th order characters. More specifically, if $\gcd(C_1 \ldots C_r, m_1 \ldots m_r) = 1$ we have

$$H_{GT}(C_1, \ldots, C_r; m_1, \ldots, m_r) = \left(\frac{m_1}{C_1}\right)^{-\|\alpha_1\|^2} \cdots \left(\frac{m_r}{C_r}\right)^{-\|\alpha_r\|^2} H_{GT}(C_1, \ldots, C_r), \tag{2}$$

where $\|\cdot\|$ is a fixed W-invariant inner product on V and $\left(\frac{\cdot}{\cdot}\right)$ is the n-th power residue symbol.

Although the coefficients $H_{GT}(C_1, \ldots, C_r; m_1, \ldots, m_r)$ in [5] are thus *roughly* twists of the original coefficients, this is only approximately true since (2) *fails* when the m_i are not coprime to the C_i. However, combining (2) together with the twisted multiplicativity of the coefficients C_i (the same rule as in (24) in Section 4 below), does allow us to reduce the specification of the coefficients to the case where the C_i and the m_i are all powers of the same prime p. In [5], this was only accomplished when Φ is of type A_r, and in that case, analytic properties could only be proved for $r \leqslant 2$, or $n = 2$ and $r \leqslant 5$, or $n = 1$. In the case of A_r, the coefficients H_{GT} of the series in [5] had the following properties.

- The existence of stable coefficients in correspondence with Weyl group elements described above for $H(p^{k_1}, \ldots, p^{k_r})$ persists, but the support in terms of k_i is changed. That is, with $m_i = p^{l_i}$ fixed and n sufficiently large, there are still $|W|$ distinct values $(k_1, \ldots, k_r)$ such that $H_{GT}(p^{k_1}, \ldots, p^{k_r}; p^{l_1}, \ldots, p^{l_r})$ is nonzero, and the coefficient corresponding to $w \in W$ is still a product of $l(w)$ Gauss sums. But when $l_i > 0$, the locations of the $(k_1, \ldots, k_r)$ parametrizing these stable coefficients form the vertices of a larger polytope than in the untwisted case. These coefficients will be called *twisted, stable* coefficients.
- If n is not sufficiently large, further nonzero coefficients appear inside the polytope whose vertices are spanned by the stable coefficients. In [5], these coefficients are described as products of Gauss sums parametrized by strict Gelfand–Tsetlin patterns. These coefficients are given a uniform description for all n, but due to the properties of Gauss sums they can vanish, and if n is sufficiently large, only the $|W|$ stable coefficients remain.

In summary, [4] contains Weyl group multiple Dirichlet series whose coefficients $H(\mathbf{c})$ are in one-to-one correspondence with elements of the Weyl group for n, the order of the power residue symbols, sufficiently large. Alternately, [5] contains multiple Dirichlet series for all n whose coefficients $H_{GT}(\mathbf{c}; \mathbf{m})$ are parametrized by Gelfand–Tsetlin patterns, and involve twists, but is restricted to Φ of Cartan type A.

In the paper at hand, the definition of the coefficients $H(\mathbf{c}, \mathbf{m})$ in Section 4 shows that

$$H(\mathbf{c}; 1, \ldots, 1) = H(\mathbf{c}),$$

the coefficients of the series in [4]. In the final section, we prove that for $\Phi = A_r$ and $n \geq r$,

$$H(\mathbf{c}; \mathbf{m}) = H_{GT}(\mathbf{c}; \mathbf{m}),$$

the coefficients of the series in [5], thereby unifying and extending the two earlier descriptions. We note that a general proof that the Weyl group multiple Dirichlet series of [5] possess functional equations (where n is not necessarily sufficiently large) remains open.

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2 Preliminaries

2.1 Weyl group action

Let V be a real vector space of dimension r containing the rank r root system Φ. Any $\alpha \in V$ may be expressed as $\alpha = \sum_{i=1}^{r} b_i \alpha_i$ for a basis of simple positive roots α_i with $b_i \in \mathbb{R}$. Then we define the pairing $B(\alpha, \mathbf{s}) : V \times \mathbb{C}^r \longrightarrow \mathbb{C}$ for $\alpha \in V$ and $\mathbf{s} = (s_1, \ldots, s_r) \in \mathbb{C}^r$ by

$$B(\alpha, \boldsymbol{s}) = \sum b_i s_i. \tag{3}$$

Note that B is just the complexification of the usual dual pairing $V \times V^\vee \longrightarrow \mathbb{R}$, but we prefer the definition above for the explicit computations of subsequent sections.

The Weyl group W of Φ has a natural action on V in terms of the pairing. For a simple reflection σ_α in a hyperplane perpendicular to α we have $\sigma_\alpha : V \longrightarrow V$ given by

$$\sigma_\alpha(x) = x - B(x, \alpha^\vee)\, \alpha,$$

where $\alpha^\vee$ is the corresponding element of the dual root system $\Phi^\vee$. In particular, the effect of σ_i on roots $\alpha \in \Phi$ is

$$\sigma_i : \alpha \mapsto \alpha - \frac{2\langle \alpha, \alpha_i \rangle}{\langle \alpha_i, \alpha_i \rangle} \alpha_i, \tag{4}$$

where $\langle \ , \ \rangle$ is the standard Euclidean inner product.

We now define a Weyl group action on $\mathbf{s} \in \mathbb{C}^r$. We will denote the image under this action by $w(\mathbf{s})$. Let $\rho^\vee$ be the Weyl vector for the dual root system, i.e. half the sum of the positive coroots. Identifying $V_{\mathbb{C}}^\vee$ with $\mathbb{C}^r$ we may take

$$\rho^\vee = (1, 1, \ldots, 1). \tag{5}$$

The action of W on $\mathbb{C}^r$ is defined implicitly according to the identification

$$B\left(w\alpha, w(\boldsymbol{s}) - \frac{1}{2}\rho^\vee\right) = B\left(\alpha, \boldsymbol{s} - \frac{1}{2}\rho^\vee\right). \tag{6}$$

For simple reflections, we have the following result ([4], Prop. 3.1).

Proposition 1. *The action of σ_i on $\boldsymbol{s} = (s_1, \ldots, s_r)$ according to (6) is given by:*

$$s_j \longmapsto s_j - \frac{2\,\langle \alpha_j, \alpha_i \rangle}{\langle \alpha_i, \alpha_i \rangle} \left(s_i - \frac{1}{2}\right), \qquad j = 1, \ldots, r. \tag{7}$$

In particular, $s_i \longmapsto 1 - s_i$. Note also that

$$-\frac{2\,\langle \alpha_j, \alpha_i \rangle}{\langle \alpha_i, \alpha_i \rangle} \geqslant 0 \quad \textit{if } j \neq i.$$

2.2 Two lemmas using root systems

In this section, we give two lemmas concerning root systems that will be used in proving local functional equations. Let Φ be a reduced root system of rank r. Recall that $\lambda \in V$ is a *weight* if $2\langle\lambda,\alpha\rangle / \langle\alpha,\alpha\rangle \in \mathbb{Z}$ for all $\alpha \in \Phi$, and the weight is *dominant* if $2\langle\lambda,\alpha\rangle / \langle\alpha,\alpha\rangle \geqslant 0$ for all $\alpha \in \Phi^+$. It is well-known that ρ is a dominant weight; in fact, it is the sum of the fundamental dominant weights ([7], Proposition 21.16).

Let $\varepsilon_1,\dots,\varepsilon_r$ be the fundamental dominant weights, which satisfy

$$\frac{2\langle\varepsilon_i,\alpha_j\rangle}{\langle\alpha_j,\alpha_j\rangle} = \delta_{ij} \quad (\delta_{ij} = \text{Kronecker delta}). \tag{8}$$

Let Λ_{weight} be the weight lattice, generated by the ε_i. It contains the root lattice Λ_{root} generated by the α_i.

We will fix nonnegative integers $l_1,\dots,l_r$ and let $\lambda = \sum l_i\varepsilon_i$ be the corresponding weight. Let Φ_w be the set of positive roots α such that $w(\alpha)$ is a negative root.

Lemma 2. *Let $w \in W$.*
(i) The cardinality of Φ_w is the length $l(w)$ of w.
(ii) Express $\rho+\lambda-w(\rho+\lambda)$ as a linear combination of the simple positive roots:

$$\rho+\lambda-w(\rho+\lambda) = \sum_{i=1}^{r} k_i\alpha_i. \tag{9}$$

Then the k_i are nonnegative integers.
(iii) If $w, w' \in W$ are such that $\rho+\lambda-w(\rho+\lambda) = \rho+\lambda-w'(\rho+\lambda)$ then $w = w'$.

Proof. Part (i) follows from Proposition 21.2 of [7]. For (ii), note that the expression (9) as an integral linear combination is valid by Proposition 21.14 of [7]. To show that this is a nonnegative linear combination, note that $\rho+\lambda$ lies inside positive Weyl chamber, as the l_i used to define λ are nonnegative. Hence, in the partial ordering, $\rho+\lambda \succ w(\rho+\lambda)$ for all $w \in W$, and the claim follows.

For (iii), we again use the fact that $\rho+\lambda$ is in the interior of the positive Weyl chamber; thus, $w(\rho+\lambda) = w'(\rho+\lambda)$ means that the positive Weyl chamber is fixed by $w^{-1}w'$, which implies $w^{-1}w' = 1$. □

Define the function d_λ on Φ^+ by

$$d_\lambda(\alpha) = \frac{2\langle\rho+\lambda,\alpha\rangle}{\langle\alpha,\alpha\rangle} = B(\rho+\lambda,\alpha^\vee). \tag{10}$$

Lemma 3. *We have $d_\lambda(\alpha) \in \mathbb{Z}^+$ for all $\alpha \in \Phi^+$, and $d_\lambda(\alpha_i) = l_i+1$ if α_i is a simple positive root.*

Proof. This follows from (8), expressing ρ as the sum of the fundamental dominant weights. □

2.3 Hilbert symbols

Let $n > 1$ be an integer and let F be a number field containing the n-th roots of unity. Let S be a finite set of places of F such that S contains all archimedean places, all places ramified over $\mathbb{Q}$, and is sufficiently large that the ring of S-integers $\mathfrak{o}_S$ is a principal ideal domain. Embed $\mathfrak{o}_S$ in $F_S = \prod_{v \in S} F_v$ diagonally.

The product of local Hilbert symbols gives rise to a pairing $(\ ,\ )_S : F_S^\times \times F_S^\times \longrightarrow \mu_n$ by $(a,b)_S = \prod_{v \in S}(a,b)_v$. A subgroup Ω of $F_S^\times$ is called *isotropic* if $(\varepsilon, \delta)_S = 1$ for $\varepsilon, \delta \in \Omega$. Let Ω be the subgroup $\mathfrak{o}_S^\times F_S^{\times,n}$, which is maximal isotropic. If t is a positive integer, let $\mathcal{M}_t(\Omega)$ be the vector space of functions $\Psi : F_{\text{fin}}^\times \longrightarrow \mathbb{C}$ that satisfy

$$\Psi(\varepsilon c) = (c, \varepsilon)_S^{-t} \Psi(c), \tag{11}$$

when $\varepsilon \in \Omega$. We denote $\mathcal{M}_1(\Omega)$ by $\mathcal{M}(\Omega)$. Note that if ε is sufficiently close to the identity in $F_{\text{fin}}^\times$, it is an n-th power at every place in S_{fin}; thus such a function is locally constant. It is easy to see that the dimension of $\mathcal{M}(\Omega)$ is $[F_S^\times : \Omega] < \infty$.

2.4 Gauss sums

If $a \in \mathfrak{o}_S$ and $\mathfrak{b}$ is an ideal of $\mathfrak{o}_S$, let $\left(\frac{a}{\mathfrak{b}}\right)$ be the n-th order power residue symbol as defined in [4]. (This depends on S, but we suppress this dependence from the notation.) If $a, c \in \mathfrak{o}_S$ and $c \neq 0$, and if t is a positive integer, define the Gauss sum $g_t(a,c)$ as follows. We choose a nontrivial additive character ψ of F_S such that $\psi(x\mathfrak{o}_S) = 1$ if and only if $x \in \mathfrak{o}_S$. (See Brubaker and Bump [3], Lemma 1.) Then the Gauss sum is given by

$$g_t(a,c) = \sum_{d \bmod c} \left(\frac{d}{c\mathfrak{o}_S}\right)^t \psi\left(\frac{ad}{c}\right). \tag{12}$$

We will also denote $g_1(a,c) = g(a,c)$.

2.5 Kubota Dirichlet series

For $\Psi \in \mathcal{M}_t(\Omega)$, the space of functions defined in (11), let

$$\mathcal{D}_t(s, \Psi, a) = \sum_{0 \neq c \in \mathfrak{o}_S / \mathfrak{o}_S^\times} g_t(a,c)\, \Psi(c)\, \mathbb{N}(c)^{-2s} \quad .$$

We will also denote $\mathcal{D}_1(s, \Psi, a) = \mathcal{D}(s, \Psi, a)$. Here $\mathbb{N}(c)$ is the order of $\mathfrak{o}_S / c\mathfrak{o}_S$. The term $g_t(a,c)\, \Psi(c)\, \mathbb{N}(c)^{-2s}$ is independent of the choice of representative c modulo S-units. It follows easily from standard estimates for Gauss sums that the series is convergent if $\Re(s) > \frac{3}{4}$.

Let

$$\boldsymbol{G}_n(s) = (2\pi)^{-2(n-1)s} n^{2ns} \prod_{j=1}^{n-1} \Gamma\left(2s - 1 + \frac{j}{n}\right). \tag{13}$$

In view of the multiplication formula for the Gamma function, we may also write

$$\boldsymbol{G}_n(s) = (2\pi)^{-(n-1)(2s-1)} \frac{\Gamma(n(2s-1))}{\Gamma(2s-1)}.$$

Let

$$\mathcal{D}_t^*(s, \Psi, a) = \boldsymbol{G}_m(s)^{[F:\mathbb{Q}]/2}\, \zeta_F(2ms - m + 1)\, \mathcal{D}_t(s, \Psi, a), \tag{14}$$

where $m = n/\gcd(n,t)$, $\frac{1}{2}[F:\mathbb{Q}]$ is the number of archimedean places of the totally complex field F, and ζ_F is the Dedekind zeta function of F.

If $v \in S_{\text{fin}}$, let q_v denote the cardinality of the residue class field $\mathfrak{o}_v/\mathfrak{p}_v$, where $\mathfrak{o}_v$ is the local ring in F_v and $\mathfrak{p}_v$ is its prime ideal. By an *S-Dirichlet polynomial* we mean a polynomial in q_v^{-s} as v runs through the finite number of places in S_{fin}.

If $\Psi \in \mathcal{M}(\Omega)$ and $\eta \in F_S^\times$, denote

$$\tilde{\Psi}_\eta(c) = (\eta, c)_S\, \Psi(c^{-1}\eta^{-1}). \tag{15}$$

It is easy to check that $\tilde{\Psi}_\eta \in \mathcal{M}(\Omega)$ and that it depends only on the class of η in $F_S^\times/F_S^{\times,n}$.

Then we have the following result, essentially proved in Brubaker and Bump [3]; see also Eckhardt and Patterson [9].

Theorem 4. *Let $\Psi \in \mathcal{M}_t(\Omega)$, and let $a \in \mathfrak{o}_S$. Let $m = n/\gcd(n,t)$. Then $\mathcal{D}_t^*(s, \Psi, a)$ has meromorphic continuation to all s, analytic except possibly at $s = \frac{1}{2} \pm \frac{1}{2m}$, where it might have simple poles. There exist S-Dirichlet polynomials $P_\eta^t(s)$ that depend only on the image of η in $F_S^\times/F_S^{\times,n}$ such that*

$$\mathcal{D}_t^*(s, \Psi, a) = \mathbb{N}(a)^{1-2s} \sum_{\eta \in F_S^\times/F_S^{\times,n}} P_{a\eta}^t(s)\mathcal{D}_t^*(1-s, \tilde{\Psi}_\eta, a). \tag{16}$$

This result, based on ideas of Kubota [12], relies on the theory of Eisenstein series. The case $t = 1$ is to be found in [3]; the general case follows as discussed in the proof of Proposition 5.2 of [4]. Importantly, the factor $\mathbb{N}(a)^{1-2s}$ does not depend on t.

2.6 Normalizing factors

As a final preliminary, we record the zeta and gamma factors that will be needed to normalize the Weyl group multiple Dirichlet series. These will be used to prove global functional equations.

Let Φ be a reduced root system of rank r, with inner product $\langle\ ,\ \rangle$ chosen such that $\|\alpha\| = \sqrt{\langle \alpha, \alpha \rangle}$ and $2\langle \alpha, \beta \rangle$ are integral for all $\alpha, \beta \in \Phi$. Let

$$n(\alpha) = \frac{n}{\gcd(n, \|\alpha\|^2)}. \tag{17}$$

If Φ is simply-laced, then we may take all roots to have length 1 and then $n(\alpha) = n$ for every $\alpha \in \Phi$. If Φ is not simply-laced but irreducible, and if $\langle\ ,\ \rangle$ is normalized so that the short roots have length 1, then

$$n(\alpha) = \begin{cases} n \text{ if } \alpha \text{ is a short root} \\ n \text{ if } \alpha \text{ is a long root and } \Phi \neq G_2, \text{ and } n \text{ is odd} \\ \frac{n}{2} \text{ if } \alpha \text{ is a long root and } \Phi \neq G_2, \text{ and } n \text{ is even} \\ n \text{ if } \alpha \text{ is a long root and } \Phi = G_2, \text{ and } 3 \nmid n \\ \frac{n}{3} \text{ if } \alpha \text{ is a long root and } \Phi = G_2, \text{ and } 3|n. \end{cases}$$

As before, if α is a positive root, write $\alpha = \sum k_i \alpha_i$. Let

$$\zeta_\alpha(\boldsymbol{s}) = \zeta_F\left(1 + 2n(\alpha)\sum_{i=1}^r k_i(s_i - \frac{1}{2})\right) = \zeta_F\left(1 + 2n(\alpha)\, B\left(\alpha, \boldsymbol{s} - \frac{1}{2}\rho^\vee\right)\right). \tag{18}$$

Also let

$$G_\alpha(\boldsymbol{s}) = \boldsymbol{G}_{n(\alpha)}\left(\frac{1}{2} + \sum_{i=1}^r k_i\left(s_i - \frac{1}{2}\right)\right) = \boldsymbol{G}_{n(\alpha)}\left(\frac{1}{2} + B\left(\alpha, \boldsymbol{s} - \frac{1}{2}\rho^\vee\right)\right),$$

where $\boldsymbol{G}_n(s)$ is defined as in (13). Define the *normalized* multiple Dirichlet series by

$$Z^*_\Psi(\boldsymbol{s}) = \left[\prod_{\alpha\in\Phi+} G_\alpha(\boldsymbol{s})\, \zeta_\alpha(\boldsymbol{s})\right] Z_\Psi(\boldsymbol{s}). \tag{19}$$

3 Stability assumption

All of our subsequent computations rely on a critical assumption that n, the order of the power residue symbols appearing in all our definitions, is sufficiently large. This dependence appears only once in the section on global functional equations, but is crucial in simplifying the proof that the multiple Dirichlet series can be understood in terms of Kubota Dirichlet series. This dependence is also crucial in making the bridge between Weyl group multiple Dirichlet series and those series defined by Gelfand–Tsetlin patterns.

Let $\sigma_i \in W$ be a fixed simple reflection about $\alpha_i \in \Phi$. Let $m_1, \ldots, m_r$ be fixed. For p a prime, let $l_i = \mathrm{ord}_p(m_i)$. (For convenience, we suppress the dependence of l_i on p in the notation.) Let

$$\lambda_p = \sum_{i=1}^r l_i \varepsilon_i. \tag{20}$$

Stability Assumption. *The positive integer n satisfies the following property. Let $\alpha = \sum_{i=1}^{r} t_i \alpha_i$ be the largest positive root in the partial ordering. Then for every prime p,*

$$n \geq \gcd(n, \|\alpha\|^2) \cdot d_{\lambda_p}(\alpha) = \gcd(n, \|\alpha\|^2) \cdot \sum_{i=1}^{r} t_i (l_i + 1) \,. \tag{21}$$

Note that the right-hand side of (21) is clearly bounded for fixed choice of $m_1, \dots, m_r$. We fix an n satisfying this assumption for the rest of the paper.

For example, if $\Phi = A_r$ and the inner product is chosen so that $\|\alpha\| = 1$ for each root α, the condition (21) becomes $n \geqslant \sum_{i=1}^{r} l_i$.

4 Definition of the twisted multiple Dirichlet series

Let $\mathcal{M}(\Omega^r)$ be as in [4], and let $\Psi \in \mathcal{M}(\Omega^r)$. In this section, we give a precise definition of the series

$$\begin{aligned} Z_\Psi(\mathbf{s}; \mathbf{m}) &= Z_\Psi(s_1, \dots, s_r; m_1, \dots, m_r) \\ &= \textstyle\sum_{\mathfrak{c}_1, \dots, \mathfrak{c}_r} H\Psi(\mathfrak{c}_1, \dots, \mathfrak{c}_r; m_1, \dots, m_r) \; \mathbb{N}\mathfrak{c}_1^{-2s_1} \cdots \mathbb{N}\mathfrak{c}_r^{-2s_r}, \end{aligned} \tag{22}$$

introduced in (1). As noted in the introduction and in [4], the product

$$H\Psi(\mathbf{c}; \mathbf{m}) = H(C_1, \dots, C_r; m_1, \dots, m_r) \, \Psi(C_1, \dots, C_r)$$

will be unchanged if C_i is multiplied by a unit, so (22) can be regarded as a sum over $C_i \in \mathfrak{o}_S / \mathfrak{o}_S^\times$ or of the ideals $\mathfrak{c}_i$ that the C_i generate.

It remains to describe the twisted coefficients H. If

$$\gcd(C_1 \cdots C_r, C_1' \cdots C_r') = 1, \tag{23}$$

then

$$\frac{H(C_1 C_1', \dots, C_r C_r'; m_1, \dots, m_r)}{H(C_1, \dots, C_r; m_1, \dots, m_r) \; H(C_1', \dots, C_r'; m_1, \dots, m_r)}$$
$$= \prod_{i=1}^{r} \left(\frac{C_i}{C_i'} \right)^{\|\alpha_i\|^2} \left(\frac{C_i'}{C_i} \right)^{\|\alpha_i\|^2} \prod_{i<j} \left(\frac{C_i}{C_j'} \right)^{2\langle \alpha_i, \alpha_j \rangle} \left(\frac{C_i'}{C_j} \right)^{2\langle \alpha_i, \alpha_j \rangle} \,. \tag{24}$$

Moreover if $\gcd(m_1' \dots m_r', C_1 \dots C_r) = 1$ we will have the multiplicativity

$$H(C_1, \dots, C_r; m_1 m_1', \dots, m_r m_r')$$
$$= \left(\frac{m_1'}{C_1} \right)^{-\|\alpha_1\|^2} \cdots \left(\frac{m_r'}{C_r} \right)^{-\|\alpha_r\|^2} H(C_1, \dots, C_r; m_1, \dots, m_r). \tag{25}$$

Equations (24) and (25) reduce the specification of general coefficients $H(C_1, \dots, C_r; m_1, \dots, m_r)$ to those of the form $H(p^{k_1}, \dots, p^{k_r}; p^{l_1}, \dots, p^{l_r})$

where p is a prime. For given $(k_1, \ldots, k_r)$, these coefficients are defined to be zero unless there exists $w \in W$ such that (9) holds. Recall that Φ_w denotes the set of all positive roots α such that $w(\alpha)$ is a negative root, and that the cardinality of Φ_w is equal to the length $l(w)$ of w in the Weyl group. If $(k_1, \ldots, k_r)$ has the property that (9) is satisfied for some $w \in W$, then we define

$$H(p^{k_1}, \ldots, p^{k_r}; p^{l_1}, \ldots, p^{l_r}) = \prod_{\alpha \in \Phi_w} g_{\|\alpha\|^2}(p^{d_{\lambda_p}(\alpha)-1}, p^{d_{\lambda_p}(\alpha)}), \tag{26}$$

where λ_p is given by (20) and $d_{\lambda_p}(\alpha) = B(\lambda_p + \rho, \alpha^\vee)$ as in (10).

5 Local computations

In this section, we analyze our multiple Dirichlet series coefficients at powers of a single fixed prime p, and show that they contain Gauss sums. These will be used to form Kubota Dirichlet series in the next section.

For the remainder of this section, let $l_1, \ldots, l_r$ be fixed nonnegative integers, and to minimize subscripting, let

$$\lambda = \lambda_p = \sum_{i=1}^{r} l_i \varepsilon_i,$$

as defined in the previous section.

We recall that on prime powers, the choices of k_i for which the expression $H(p^{k_1}, \ldots, p^{k_r}; p^{l_1}, \ldots, p^{l_r})$ is nonzero are in one-to-one correspondence with elements $w \in W$, the Weyl group. We say that $(k_1, \ldots, k_r) \in \mathbb{Z}^r$ is *associated to $w \in W$ with respect to λ* if (9) is satisfied; in this case, we write

$$(k_1, \ldots, k_r) = \mathrm{assoc}_\lambda(w).$$

The following results are generalizations of Propositions 4.1 and 4.2 of [4].

Proposition 5. *Let $w \in W$ be such that $l(\sigma_i w) = l(w) + 1$. Suppose that* $\mathrm{assoc}_\lambda(w) = (k_1, \ldots, k_r) \in \mathbb{Z}^r$ *and* $\mathrm{assoc}_\lambda(\sigma_i w) = (h_1, \ldots, h_r)$. *Let $d_\lambda = d_\lambda(w^{-1}\alpha_i)$ in the notation (10). Then*

$$h_j = \begin{cases} k_i + d_\lambda & \text{if } j = i \\ k_i & \text{if } j \neq i, \end{cases} \tag{27}$$

and

$$H(p^{h_1}, \ldots, p^{h_r}) = g_{\|\alpha_i\|^2}(p^{d_\lambda - 1}, p^{d_\lambda})\, H(p^{k_1}, \ldots, p^{k_r}). \tag{28}$$

Proof. This is proved similarly to Proposition 4.1 of [4], by replacing ρ with $\rho + \lambda$ and d with d_λ. ☐

Proposition 6. *Let $d_\lambda = d_\lambda(w^{-1}(\alpha_i))$ and let $l_1, \ldots, l_r$ be fixed as above. For any $w \in W$, the monomial in the r complex variables $\boldsymbol{s} = (s_1, \ldots, s_r)$*

$$\mathbb{N}p^{(s_i - \frac{1}{2})(d_\lambda - l_i - 1)} \prod_{\alpha \in \Phi_w} \mathbb{N}p^{-2B(\rho + \lambda - w(\rho+\lambda), \mathbf{s})}$$

is invariant under the action of σ_i given in (7).

Proof. The statement is equivalent to showing that

$$\frac{1}{2}(d_\lambda - l_i - 1)\alpha_i + \rho + \lambda - w(\rho + \lambda) \tag{29}$$

is orthogonal to α_i. Hence, it suffices to show (29) is fixed by σ_i, i.e.,

$$\sigma_i(\rho + \lambda) - \sigma_i w(\rho + \lambda) = (d_\lambda - l_i - 1)\alpha_i + (\rho + \lambda) - w(\rho + \lambda). \tag{30}$$

Since $\rho + \lambda - \sigma_i(\rho + \lambda) = (1 + l_i)\alpha_i$, we can write (30) as $w(\rho+\lambda) - \sigma_i w(\rho+\lambda) = d_\lambda \alpha_i$; and indeed, applying the definition given by (10)

$$d_\lambda \alpha_i = \frac{2\left\langle w^{-1}\alpha_i, \rho + \lambda \right\rangle}{\langle \alpha_i, \alpha_i \rangle}\alpha_i = \frac{2\left\langle \alpha_i, w(\rho + \lambda) \right\rangle}{\langle \alpha_i, \alpha_i \rangle}\alpha_i = w(\rho + \lambda) - \sigma_i w(\rho + \lambda).$$

□

6 Preparing the global Dirichlet series

We preserve the notations above. In particular, $m_1, \ldots, m_r$ are fixed integers with corresponding λ_p defined for each prime p as in (20). In our local computations, we showed a connection between the prime-power coefficients $H(p^{k_1}, \ldots, p^{k_r})$ associated to pairs of Weyl group elements w and $\sigma_i w$ for a fixed simple reflection σ_i and Gauss sums. The next step is to translate this into a global notion. Once the correct definitions are given, it turns out to be relatively straightforward to generalize the proofs in [4], so we will omit many proof details which follow by very similar methods to [4].

In [4], the notion of admissibility for r-tuples of integers $(C_1, \ldots, C_r)$ in $(\mathfrak{o}_S)^r$ was defined. We generalize this in the following definition.

Definition 7. *We say that $(C_1, \ldots, C_r)$ in $(\mathfrak{o}_S)^r$ is* admissible with respect to $\mathbf{m}$ *if, for each prime p, there exists a Weyl group element $w_p \in W$ such that*

$$(\mathrm{ord}_p(C_1), \ldots, \mathrm{ord}_p(C_r)) = \mathrm{assoc}_{\lambda_p}(w_p).$$

For such $(C_1, \ldots, C_r)$, we say that C_i is i-reduced *if, for every p, we have $l(\sigma_i w_p) = l(w_p) + 1$.*

We note that if $C_1, \ldots, C_r$ are nonzero elements of $\mathfrak{o}_S$, then $\mathbf{c}$ is admissible with respect to $\mathbf{m}$ if and only if $H(\mathbf{c};\mathbf{m}) \neq 0$. This is immediate from the definition of H.

We have the following results.

Proposition 8. *Let $C_1, \ldots, C_{i-1}, C_{i+1}, \ldots, C_r$ be nonzero elements of $\mathfrak{o}_S$. If there exists a C_i such that $(C_1, \ldots, C_r)$ is admissible with respect to λ, then there exists a C_i' (modulo the action of $\mathfrak{o}_S^\times$) that is i-reduced. This C_i' divides C_i and is uniquely determined up to multiplication by a unit. Moreover, for each prime p, if w_p' is determined by the equality*

$$(\mathrm{ord}_p(C_1), \ldots, \mathrm{ord}_p(C_i'), \ldots, \mathrm{ord}_p(C_r)) = (k_1, \ldots, k_r) = \mathrm{assoc}_{\lambda_p}(w_p'),$$

then either $\mathrm{ord}_p(C_i) = k_i$ or $\mathrm{ord}_p(C_i) = k_i + d_\lambda$, where $d_\lambda = d_\lambda((w_p')^{-1}\alpha_i)$.

Proof. The proof is similar to Proposition 5.2 of [4], replacing "admissible" with "admissible with respect to λ" and d with d_λ. □

The multiple Dirichlet series is built out of H-coefficients of the form $H(C_1, \ldots, C_r; m_1, \ldots, m_r)$, satisfying the multiplicativity relation (25). However, for convenience we will suppress the m_i's from the notation. By Proposition 8, $Z_\Psi(s_1, \ldots, s_r) =$

$$\sum_{\substack{0 \neq C_j \in \mathfrak{o}_S^\times \backslash \mathfrak{o}_S \\ 1 \leqslant j \leqslant r \\ C_1, \ldots, C_r \text{ admissible w.r.t. } \lambda \\ C_i \ i\text{-reduced}}} \mathbb{N}C_1^{-2s_1} \ldots \mathbb{N}C_r^{-2s_r} H(C_1, \ldots, C_r) \sum_{0 \neq D \in \mathfrak{o}_S^\times \backslash \mathfrak{o}_S} (D, C_i)_S^{\|\alpha_i\|^2}$$
$$\times \frac{H(C_1, C_2, \ldots, DC_i, \ldots, C_r)}{H(C_1, C_2, \ldots, C_i, \ldots, C_r)} \prod_{j>i} (D, C_j)_S^{2\langle \alpha_i, \alpha_j \rangle} \Psi_i^{C_1, \ldots, C_r}(D)\, \mathbb{N}D^{-2s_i}, \quad (31)$$

where we define

$$\Psi_i^{C_1, \ldots, C_r}(D) = \Psi(C_1, \ldots, C_i D, \ldots, C_r)(D, C_i)_S^{-\|\alpha_i\|^2} \prod_{j>i} (D, C_j)_S^{-2\langle \alpha_i, \alpha_j \rangle} \quad (32)$$

to emphasize the dependence on D for fixed parameters $C_1, \ldots, C_r$ in the inner sum.

We recall Lemma 5.3 of [4].

Lemma 9 ([4], Lemma 5.3). *Let $C_1, \ldots, C_r$ be fixed nonzero elements of $\mathfrak{o}_S$. Then with the notation (32), the function $\Psi_i^{C_1, \ldots, C_r} \in \mathcal{M}_{\|\alpha_i\|^2}(\Omega)$.*

One can now show that the inner sum in (31) is a Kubota Dirichlet series. The key is to identify the quotient of H's and Hilbert symbols in (31) as a Gauss sum. If the C_i and D are powers of a single prime p, this expression is (28), which is generalized below in (34).

Lemma 10. *Fix an integer* $i \in \{1, \ldots, r\}$ *and integers* $(m_1, \ldots, m_r)$*. If* $(C_1, \ldots, C_r) \in \mathfrak{o}_S^r$ *is admissible with respect to* λ *with* C_i *i-reduced, then*

$$B_i = \prod_{j=1}^{r} C_j^{-2\langle \alpha_j, \alpha_i \rangle / \langle \alpha_i, \alpha_i \rangle} \tag{33}$$

is an $\mathfrak{o}_S$ *integer and for every* $D \in \mathfrak{o}_S$ *we have*

$$\frac{H(C_1, \ldots, DC_i, \ldots, C_r)}{H(C_1, \ldots, C_r)} (D, C_i)_S^{\|\alpha_i\|^2} \prod_{j>i} (D, C_j)_S^{2\langle \alpha_i, \alpha_j \rangle} = g_{\|\alpha_i\|^2}(m_i B_i, D). \tag{34}$$

Moreover for each prime p *of* $\mathfrak{o}_S$ *we have*

$$\mathrm{ord}_p(B_i) = d_{\lambda_p}(w_p^{-1}\alpha_i) - l_i - 1, \tag{35}$$

where λ_p *corresponds to the* m_j *as in (20), and* w_p *is determined by the condition*

$$\mathrm{assoc}_{\lambda_p}(w_p) = (\mathrm{ord}_p(C_1), \ldots, \mathrm{ord}_p(C_r)).$$

The proof of this is similar to Lemma 5.3 of [4], with modifications similar to those above, and is therefore omitted.

Using Lemmas 9 and 10, we may rewrite the Dirichlet series $Z_\Psi(s_1, \ldots, s_r)$ in terms of a Kubota Dirichlet series in the variable s_i.

Proposition 11. *With notations as above, we have*

$$Z_\Psi(s_1, \ldots, s_r)$$

$$= \sum_{\substack{0 \neq C_j \in \mathfrak{o}_S/\mathfrak{o}_S^\times \\ (C_1, \ldots, C_r) \text{ admissible w.r.t. } \lambda \\ C_i \text{ } i\text{-reduced}}} \mathbb{N}C_1^{-2s_1} \ldots \mathbb{N}C_r^{-2s_r} H(C_1, \ldots, C_r)\, \mathcal{D}_{\|\alpha_i\|^2}(s_i, \Psi_i^{C_1, \ldots, C_r}, m_i B_i),$$

where, for fixed $C_1, \ldots, C_r$*, the coefficient* B_i *is defined in (33).*

Proof. We have already rewritten the Dirichlet series $Z_\Psi(s_1, \ldots, s_r)$ in equation (31) in terms of sums over C_j, $j = 1, \ldots, r$ with C_i i-reduced. The proposition then follows immediately from the previous two lemmas and the definition of $\mathcal{D}_t(s, \Psi, C)$ for S-integer C and $\Psi \in \mathcal{M}_t(\Omega)$, where $t = \|\alpha_i\|^2$. □

7 Global functional equations

Using Proposition 11 as our starting point, we are finally ready to prove functional equations corresponding to the transformations σ_i defined in (7), for each $i = 1, \ldots, r$. First we recall some notation from [4].

Let $\mathcal{A}$ be the ring of (Dirichlet) polynomials in $q_v^{\pm 2s_1}, \ldots, q_v^{\pm 2s_r}$ where v runs through the finite set of places S_{fin}, and let $\mathfrak{M} = \mathcal{A} \otimes \mathcal{M}(\Omega^r)$. We may regard elements of $\mathfrak{M}$ as functions $\Psi : \mathbb{C}^r \times (F_S^\times)^r \longrightarrow \mathbb{C}$ such that for any fixed $\mathbf{s} = (s_1, \ldots, s_r) \in \mathbb{C}^r$ the function

$$\mathbf{c} \longmapsto \Psi(\mathbf{s}; \mathbf{c})$$

defines an element of $\mathcal{M}(\Omega^r)$, while for any $\mathbf{c} \in (F_S^\times)^r$, the function

$$\mathbf{s} \longmapsto \Psi(\mathbf{s}; \mathbf{c})$$

is an element of $\mathcal{A}$. We will sometimes use the notation

$$\Psi_{\mathbf{s}}(C_1, \ldots, C_r) = \Psi(\mathbf{s}; \mathbf{c}), \qquad \boldsymbol{s} \in \mathbb{C}^r. \tag{36}$$

We identify $\mathcal{M}(\Omega^r)$ with its image $1 \otimes \mathcal{M}(\Omega^r)$ in $\mathfrak{M}$; this just consists of the $\Psi_{\boldsymbol{s}}$ that are independent of $\boldsymbol{s} \in \mathbb{C}^r$.

The operators σ_i on $\mathbb{C}^r$ are defined in (7). Define corresponding operators σ_i on $\mathfrak{M}$ by

$$(\sigma_i \Psi_{\boldsymbol{s}})(\mathbf{c}) = (\sigma_i \Psi)(\mathbf{s}; C_1, \ldots, C_r) = \tag{37}$$

$$\sum_{\eta \in F_S^\times / F_S^{\times, n}} (\eta, C_i)_S^{\|\alpha_i\|^2} \prod_{j>i} (\eta, C_j^{2\langle \alpha_i, \alpha_j \rangle})_S P_{\eta m_i B_i}(s_i) \Psi(\sigma_i(\mathbf{s}); C_1, C_2, \ldots, \eta^{-1} C_i, \ldots, C_r)$$

where, as in (33),

$$B_i = \prod_j C_j^{-2\langle \alpha_i, \alpha_j \rangle / \langle \alpha_i, \alpha_i \rangle} = C_i^{-2} \prod_{j \neq i} C_j^{-2\langle \alpha_i, \alpha_j \rangle / \langle \alpha_i, \alpha_i \rangle}.$$

We have arranged this definition to give a clean formulation of the functional equations. Note that the Dirichlet polynomials P are associated to n-th power classes which depend on the fixed parameter m_i in $(m_1, \ldots, m_r)$, though we suppress this from the notation for the action σ_i on $\mathfrak{M}$.

Proposition 12. *If $\Psi \in \mathfrak{M}$, then $\sigma_i \Psi$ is in $\mathfrak{M}$.*

Proof. See [4], Prop. 8, for a proof. (Replace all instances of C_0 there with $m_i B_i$ to obtain the present result.) □

Each functional equation corresponding to $\sigma_i \in W$ is inherited from a functional equation for the Kubota Dirichlet series appearing in Proposition 11. These functional equations are formalized in the following result.

Lemma 13. *Given an element $\Psi_{\boldsymbol{s}}(C_1, \ldots, C_r) \in \mathfrak{M}$, we have*

$$\mathcal{D}^*_{\|\alpha_i\|^2}(s_i, \Psi_i^{C_1, \ldots, C_r}, m_i B_i) = \mathbb{N}(m_i B_i)^{1-2s_i} \mathcal{D}^*_{\|\alpha_i\|^2}(1 - s_i, (\sigma_i \Psi)_i^{C_1, \ldots, C_r}, m_i B_i),$$

where $\mathcal{D}^*_{\|\alpha_i\|^2}$ is as in (14).

Proof. This follows from (16). To check the way in which the Ψ function changes under the functional equation, apply the definition in (37) as in Lemma 5.7 of [4], with the substitution $C_0 = m_i B_i$ in every instance it appears. □

Let $\mathfrak{W}$ denote the group of automorphisms of $\mathfrak{M}$ generated by σ_i. This will turn out to be the group of functional equations for the multiple Dirichlet series. The natural homomorphism $\mathfrak{W} \longrightarrow W$ gives an action of $\mathfrak{W}$ on $\mathbb{C}^r$ induced by the action of W, and if $w \in \mathfrak{W}$, we will denote by $w\boldsymbol{s}$ the effect of w on $\boldsymbol{s} \in \mathbb{C}^r$ in this induced action. Further, recall the definition of $n(\alpha)$ for $\alpha \in \Phi$ given in (17) by

$$n(\alpha) = \frac{n}{\gcd(n, \|\alpha\|^2)}.$$

Theorem 14. *The function $Z_\Psi^*(\mathbf{s};\mathbf{m})$ has meromorphic continuation in $\mathbf{s}$ to the complex space $\mathbb{C}^r$. Moreover, for each $w \in \mathfrak{W}$ we may identify w with its image in the Weyl group and writing $w = \sigma_{j_1} \dots \sigma_{j_k}$ as a product of simple reflections, $Z_\Psi^*(\mathbf{s};\mathbf{m}) = Z_\Psi^*(\mathbf{s}; m_1, \dots, m_r)$ satisfies the functional equation*

$$Z_\Psi^*(\mathbf{s};\mathbf{m}) = \prod_{i=1}^{k} m_{j_i}^{1-2(\sigma_{j_1}\dots\sigma_{j_{i-1}})(s_{j_i})} Z_{w\Psi}^*(w\mathbf{s};\mathbf{m}), \tag{38}$$

where the action of w on $\mathfrak{M}$ is similarly given by the composition of simple reflections σ_i. It is analytic except along the hyperplanes $B(\alpha; \mathbf{s} - \frac{1}{2}\rho^\vee) = \frac{1}{2n(\alpha)}$, where α runs through Φ, $\frac{1}{2}\rho^\vee = \left(\frac{1}{2}, \dots, \frac{1}{2}\right)$, and B is defined by (3); along these hyperplanes it can have simple poles.

Observe that the equation $B(-\alpha; \mathbf{s} - \frac{1}{2}\rho^\vee) = \frac{1}{2n(\alpha)}$ is equivalent to $B(\alpha; \mathbf{s} - \frac{1}{2}\rho^\vee) = -\frac{1}{2n(\alpha)}$, so the polar hyperplanes occur in parallel pairs.

Proof. The proof is based on Bochner's tube domain theorem. We sketch this argument below, and refer the reader to [2] where the case $\Phi = A_2$ is worked out in full detail, and to [4] where a similar argument to the one below is applied for $(m_1, \dots, m_r) = (1, \dots, 1)$.

From the standard estimates for the Gauss sums and Proposition 11, it follows that the original Dirichlet series defining Z_Ψ is absolutely convergent in a translate of the fundamental Weyl chamber, denoted

$$\Lambda_0 = \left\{ \boldsymbol{s} = (s_1, \dots, s_r) \quad | \quad \Re(s_j) > \frac{3}{4}, \quad j = 1, \dots, r \right\}.$$

Using standard growth estimates for the Kubota Dirichlet series, one sees that the expression in Proposition 11 is analytic in the convex hull of $\Lambda_0 \cup \sigma_i \Lambda_0$, which we will denote by Λ_i. On this region, we claim that for the simple reflection σ_i one has

$$Z_{\sigma_i\Psi}^*(\sigma_i \boldsymbol{s}) = \mathbb{N}(m_i)^{1-2s_i} Z_\Psi^*(\boldsymbol{s}).$$

Recalling the effect of the transformation σ_i on $\mathbb{C}^r$ from (7), we note that the Kubota Dirichlet series appearing in Proposition 11 is essentially invariant under this transformation by Lemma 13, with the implicit Ψ function mapped to the appropriately defined $\sigma_i\Psi$. That is, the Kubota Dirichlet series functional equation produces a factor of $\mathbb{N}(m_i B_i)^{1-2s_i}$. Thus, it remains to show that for each of the r-tuples $(C_1, \ldots, C_r)$ that is admissible with respect to λ with C_i i-reduced, the corresponding terms on the right-hand side of Proposition 11, multiplied by $\mathbb{N}(m_i B_i)^{1/2-s_i}$ are invariant under $\boldsymbol{s} \mapsto \sigma_i(\boldsymbol{s})$ given explicitly in (7). This follows from Proposition 6, since with our definitions, when $C_1, \ldots, C_r$ is admissible with respect to λ and C_i is i-reduced,

$$\mathbb{N}C_1^{-2s_1} \cdots \mathbb{N}C_r^{-2s_r} = \prod_p \prod_{\alpha \in \Phi_w} \mathbb{N}p^{-2B(\rho+\lambda-w(\rho+\lambda),\mathbf{s})}$$

and, by (35)

$$\mathbb{N}B_i = \mathbb{N}m_i^{-1} \prod_p \mathbb{N}p^{d_{\lambda_p}(w_p^{-1}\alpha_i)-1}.$$

Finally, regarding the normalizing factor, one factor $G_{\alpha_i}(\boldsymbol{s})\zeta_{\alpha_i}(\boldsymbol{s})$ from (19) is needed to normalize $\mathcal{D}^*_{\|\alpha_i\|^2}(s_i, \Psi_i^{C_1,\ldots,C_r}, C_0)$ in Proposition 11; the remaining factors are permuted amongst themselves since σ_i permutes $\Phi^+ - \{\alpha_i\}$.

Arguing as in [2] we obtain analytic continuation to any simply-connected region Λ' that is a union of W-translates of the Λ_i obtained by composing functional equations. We may choose Λ' so that its convex hull is all of $\mathbb{C}^r$. The meromorphic continuation to all of $\mathbb{C}^r$ follows from Bochner's tube domain theorem (Bochner [1] or Hörmander [10], Theorem 2.5.10). As in [2], one actually applies Bochner's theorem to the function

$$Z_\Psi(\boldsymbol{s}) \prod_{\alpha \in \Phi} \left(B\left(\alpha; \boldsymbol{s} - \frac{1}{2}\rho^\vee\right) - \frac{1}{2n(\alpha)} \right),$$

since inclusion of the factors to cancel the poles of Z_Ψ gives a function that is everywhere analytic. □

We have not proved that the natural map $\mathfrak{W} \longrightarrow W$ is an isomorphism. It is highly likely that this is true, but not too important for the functional equations as we will now see. We recall from the introduction that we defined $\mathfrak{M}'$ to be the quotient of $\mathfrak{M}$ by the kernel of the map $\Psi \longmapsto Z_\Psi$.

Corollary 15. *If $w \in \mathfrak{W}$ is in the kernel of the map $\mathfrak{W} \longrightarrow W$ then $Z_\Psi = Z_{w\Psi}$. Thus there is an action of W on $\mathfrak{M}'$ that is compatible with the action of $\mathfrak{W}$ on $\mathfrak{M}$.*

Proof. Since such a w acts trivially on $\mathbb{C}^r$, (38) implies that Z_Ψ and $Z_{w\Psi}$ are multiple Dirichlet series that agree in the region of absolute convergence. Hence they are equal. □

Thus Z_Ψ can be regarded as depending on $\Psi \in \mathfrak{M}'$, on which W acts, and so W can truly be regarded as the group of functional equations of the Weyl group multiple Dirichlet series. This completes the generalization of results from [4].

8 The Gelfand–Tsetlin pattern conjecture

In this section, we turn to the case $\Phi = A_r$ that was investigated in [5]. In particular, $\|\alpha\| = 1$ for all roots α. We will then show that the multiple Dirichlet series constructed above (now known to have a group of functional equations according to Theorem 14) are in fact the same as the multiple Dirichlet series given in [5] via a combinatorial prescription in terms of strict Gelfand–Tsetlin patterns.

Let us recall the description of these Weyl group multiple Dirichlet series in [5]. Recall that a Gelfand–Tsetlin pattern is a triangular array of integers

$$\mathfrak{T} = \left\{ \begin{array}{ccccccc} a_{00} & & a_{01} & & a_{02} & \cdots & & a_{0r} \\ & a_{11} & & a_{12} & & & a_{1r} & \\ & & \ddots & & & \iddots & & \\ & & & & a_{rr} & & & \end{array} \right\} \tag{39}$$

where the rows interleave; that is, $a_{i-1,j-1} \geqslant a_{i,j} \geqslant a_{i-1,j}$. The pattern is *strict* if each row is strictly decreasing. The strict Gelfand–Tsetlin pattern $\mathfrak{T}$ in (39) is *left-leaning* at (i,j) if $a_{i,j} = a_{i-1,j-1}$, *right-leaning* at (i,j) if $a_{i,j} = a_{i-1,j}$, and *special* at (i,j) if $a_{i-1,j-1} > a_{i,j} > a_{i-1,j}$.

Given a strict Gelfand–Tsetlin pattern, for $j \geq i$ let

$$s_{ij} = \sum_{k=j}^{r} a_{ik} - \sum_{k=j}^{r} a_{i-1,k}, \tag{40}$$

and define

$$\gamma(i,j) = \begin{cases} \mathbb{N}p^{s_{ij}} & \text{if } \mathfrak{T} \text{ is right-leaning at } (i,j) \\ g(p^{s_{ij}-1}, p^{s_{ij}}) & \text{if } \mathfrak{T} \text{ is left-leaning at } (i,j) \\ \mathbb{N}p^{s_{ij}}(1-\mathbb{N}p^{-1}) & \text{if } (i,j) \text{ is special and } n | s_{ij} \\ 0 & \text{if } (i,j) \text{ is special and } n \nmid s_{ij}. \end{cases}$$

Also, define

$$G(\mathfrak{T}) = \prod_{j \geqslant i \geqslant 1} \gamma(i,j). \tag{41}$$

Given nonnegative integers k_i, l_i, $1 \leq i \leq r$, and a prime p, we define the p-th contribution to the coefficient of a multiple Dirichlet series by

$$H_{GT}(p^{k_1}, \ldots, p^{k_r}; p^{l_1}, \ldots, p^{l_r}) = \sum_{\mathfrak{T}} G(\mathfrak{T}) \tag{42}$$

where the sum is over all strict Gelfand–Tsetlin patterns $\mathfrak{T}$ with top row

$$l_1 + \cdots + l_r + r, l_2 + \cdots + l_r + r - 1, \ldots, l_r + 1, 0$$

such that for each i, $1 \le i \le r$,

$$\sum_{j=i}^{r} (a_{ij} - a_{0,j}) = k_i. \tag{43}$$

Note that $(k_1, \ldots, k_r) = k(\mathfrak{T})$ in the notation of [5]. The general coefficient of the multiple Dirichlet series, $H_{GT}(C_1, \ldots, C_r; m_1, \ldots, m_r)$, is then defined by means of twisted multiplicativity as in (24) and (25). In [5] we conjecture that these multiple Dirichlet series have meromorphic continuation and satisfy functional equations. We prove this below for n satisfying the Stability Assumption.

We begin by relating the Stability Assumption to the Gelfand–Tsetlin patterns. We recall from [5] that a strict Gelfand–Tsetlin pattern is *stable* if every entry equals one of the two directly above it (unless, of course, it is in the top row). If the top row is fixed, there are $(r+1)!$ strict stable patterns.

Proposition 16. *Suppose that the Stability Assumption (21) holds. If $\mathfrak{T}$ appearing in the sum (42) is not stable, then $G(\mathfrak{T}) = 0$.*

Proof. Suppose that $\mathfrak{T}$ is special at (i, j) and that (21) holds. Recall that s_{ij} is given by (40). Since $a_{i,k} \ge a_{i-1,k}$ for all $k \ge i$ and $a_{i,j} > a_{i-1,j}$ it follows that $s_{ij} > 0$. Similarly, since $a_{i,k} \le a_{i-1,k-1}$ for all $k \ge i$ and $a_{i,j} < a_{i-1,j-1}$, it follows that

$$s_{ij} = \sum_{k=j}^{r} a_{ik} - \sum_{k=j}^{r} a_{i-1,k} < \sum_{k=j}^{r} a_{i-1,k-1} - \sum_{k=j}^{r} a_{i-1,k} = a_{i-1,j-1} - a_{i-1,r}.$$

Since each entry of $\mathfrak{T}$ is at most $l_1 + \cdots + l_r + r$, it follows that $s_{i,j} < n$. But then $0 < s_{ij} < n$, and this implies that n does not divide s_{ij}. Hence $\gamma(i, j) = 0$, and $G(\mathfrak{T}) = 0$, as claimed. □

We identify the A_r root system with the set of vectors $\mathrm{e}_i - \mathrm{e}_j$ with $i \ne j$ where

$$\mathrm{e}_i = (0, \ldots, 1, \ldots, 0) \in \mathbb{R}^{r+1}, \qquad 1 \text{ in the } i\text{-th position.}$$

The simple positive roots are $\alpha_i = \mathrm{e}_i - \mathrm{e}_{i+1}$. The root system lies in the hyperplane V of $\mathbb{R}^{r+1}$ orthogonal to the vector $\sum \mathrm{e}_i$. Particularly $\rho = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha$ and the fundamental dominant weights ε_i are given by

$$\rho = \left(\frac{r}{2}, \frac{r}{2} - 1, \ldots, -\frac{r}{2} \right), \qquad \varepsilon_i = (1, \ldots, 1, 0, \ldots, 0) - \frac{i}{r}(1, \ldots, 1),$$

where in the definition of ε_i there are i 1's in the first vector. The action of $W = S_{r+1}$ on vectors in $\mathbb{R}^{r+1}$ is by

$$w(t_1, t_2, t_3, \ldots, t_{r+1}) = (t_{w^{-1}(1)}, t_{w^{-1}(2)}, t_{w^{-1}(3)}, t_{w^{-1}(4)}, \ldots, t_{w^{-1}(r+1)}).$$

Suppose that $\mathfrak{T}$ is a strict Gelfand–Tsetlin pattern with $a_{0r} = 0$. We may find nonnegative integers l_i so that the top row of the pattern is

$$l(r), \ldots, l(0) \tag{44}$$

with

$$l(k) = k + l_{r-k+1} + \cdots + l_r, \qquad 1 \leqslant k \leqslant r, \qquad l(0) = 0.$$

Thus $a_{0j} = l(r-j)$. Let $\lambda = \sum l_i \varepsilon_i$. We call λ the *dominant weight associated with* $\mathfrak{T}$. We will associate a Weyl group element $w \in W$ with each $\mathfrak{T}$ that is stable in the next result.

Proposition 17. *Let $\mathfrak{T}$ be a stable strict Gelfand–Tsetlin pattern with $a_{0r} = 0$, and with associated dominant weight vector λ. Define nonnegative integers $k_1, \ldots, k_r$ by (43) and also let $k_{r+1} = k_0 = 0$. Then there exists a unique element $w \in W = S_{r+1}$ such that*

$$\rho + \lambda - w(\rho + \lambda) = (k_1, k_2 - k_1, k_3 - k_2, \ldots, -k_r) = \sum k_i \alpha_i. \tag{45}$$

In fact, for $0 \leqslant i \leqslant r$

$$k_i - k_{i+1} + l(r-i) = l(r + 1 - w^{-1}(i+1)) \tag{46}$$

is the unique element in the i-th row that is not in the $(i+1)$-th row. We have

$$\begin{aligned} &\rho + \lambda - w(\rho + \lambda) \\ &= (l(r) - l(r+1-w^{-1}(1)), \ldots, l(0) - l(r+1-w^{-1}(r+1))). \end{aligned} \tag{47}$$

Proof. Let R be the top row (44) of $\mathfrak{T}$. With $\lambda = \sum l_i \varepsilon_i$, the vector $\rho + \lambda$ differs from the top row (44) by a multiple of $(1, \ldots, 1)$, which is canceled away when we compute $\rho + \lambda - w(\rho + \lambda)$. Thus it is equivalent to show that $R - w(R) = \sum k_i \alpha_i$ for a unique permutation $w \in S_{r+1}$, or in other words, that $R - \sum k_i \alpha_i$ is a permutation of R. By (43)

$$k_i - k_{i+1} + l(r-i) = k_i - k_{i+1} + a_{0,i} = \left(\sum_{j=i}^{r} a_{i,j} \right) - \left(\sum_{j=i+1}^{r} a_{i+1,j} \right).$$

Remembering the pattern is stable, the terms in the second sum may all be found in the first sum, so

$$k_i - k_{i+1} + l(r-i) = a_{ij},$$

where a_{ij} is the unique element of the i-th row that is not in the $(i+1)$-th row. Now (47) and (46) are also clear. □

We now develop some facts necessary to compare the Gelfand–Tsetlin multiple Dirichlet series to the Weyl group multiple Dirichlet series in the twisted stable case.

Lemma 18. *Let $\mathfrak{T}$ be a stable strict Gelfand–Tsetlin pattern with $a_{0r} = 0$. For $j \geq 1$, we have*

$$\begin{aligned}
&\{a_{jj}, a_{j,j+1}, \ldots, a_{j,r}\} \\
&= \{k_j - k_{j+1} + l(r-j), k_{j+1} - k_{j+2} + l(r-j-1), \ldots, k_{r-1} - k_r + l(1), k_r + l(0)\} \\
&= \{l(r+1-w^{-1}(j+1)), l(r+1-w^{-1}(j+2)), \ldots, l(r+1-w^{-1}(r+1))\}.
\end{aligned}$$

Proof. The statement follows by induction from the fact that $k_i - k_{i+1} + l(r-i)$ is the unique element of the i-th row that is not in the $(i-1)$-th row. □

For $w \in S_{r+1}$, an *i-inversion* is a j such that $i < j \leq r+1$ but $w(i) > w(j)$.

Proposition 19. *Let $\mathfrak{T}$ be a stable strict Gelfand–Tsetlin pattern with $a_{0r} = 0$, and let $w \in S_{r+1}$ be associated to $\mathfrak{T}$ as in Proposition 17. Then the number of i-inversions of w^{-1} equals the number of left-leaning entries in the i-th row of $\mathfrak{T}$.*

For example, let w^{-1} be the permutation (143), and take $\lambda = 0$. We find that $\rho - w(\rho) = (3, 0, -2, -1) = 3\alpha_1 + 3\alpha_2 + \alpha_3$ and so the corresponding Gelfand–Tsetlin pattern is the unique pattern with $(k_1, k_2, k_3) = (3, 3, 1)$. This pattern is

$$\mathfrak{T} = \left\{ \begin{array}{ccccccc}
3 & & 2 & & 1 & & 0 \\
& \boxed{3} & & \boxed{2} & & \boxed{1} & \\
& & \boxed{3} & & 1 & & \\
& & & 1 & & &
\end{array} \right\}$$

where we have marked the location of the left-leaning entries. The number of i-inversions of w^{-1} is:

i	i-inversions of w^{-1}	number
1	$(1,2), (1,3), (1,4)$	3
2	$(2,3)$	1
3	none	0

As Proposition 19 states, the number of i-inversions of w^{-1} determines the number of left-leaning entries in the i-th row; since the i-inversions are obviously forced to the left in a stable pattern, this number is also the location of the last left-leaning entry.

Proof. The i-th row, together with the rows immediately above and below, are:

$$\begin{array}{ccccccccccc}
a_{i-1,i-1} & & a_{i-1,i} & & a_{i-1,i+1} & & \cdots & & a_{i-1,r-1} & & a_{i-1,r} \\
& a_{i,i} & & a_{i,i+1} & & \cdots & & a_{i,r-1} & & a_{i,r} & \\
& & a_{i+1,i+1} & & a_{i+1,i+2} & & \cdots & & a_{i+1,r} & &
\end{array}$$

If we assume that there are exactly m left-leaning entries in the i-th row, then

$$a_{i,i} = a_{i-1,i-1}, \ldots, a_{i,i+m-1} = a_{i-1,i-2+m} \qquad (48)$$

while

$$a_{i,i+m} = a_{i-1,i+m}, \ldots, a_{i,r} = a_{i-1,r}. \tag{49}$$

The number of i-inversions of w^{-1} is the number of elements of the set

$$\{w^{-1}(i+1), \ldots, w^{-1}(r+1)\}$$

that are less than $w^{-1}(i)$. Since the function l is monotone, this equals the number of elements of the set

$$\big\{l(r+1-w^{-1}(i+1)), \ldots, l(r+1-w^{-1}(r+1))\big\} \tag{50}$$

that are greater than $l(r+1-w^{-1}(i))$. By Lemma 18, the numbers in the set (50) are just the elements of the i-th row of $\mathfrak{T}$, and $l(r+1-w^{-1}(i))$ is the unique element of the $(i-1)$-th row that doesn't occur in the i-th row. Thus the elements of the i-th row that are greater than $l(r+1-w^{-1}(i))$ are precisely the left-leaning entries in the row. □

Theorem 20. *Suppose that $\Phi = A_r$ and that $\langle \ , \ \rangle$ is chosen so that $\|\alpha\| = 1$ for all $\alpha \in \Phi$. Suppose also that the Stability Assumption (21) holds.*
(i) Let $\mathfrak{T}$ be a stable strict Gelfand–Tsetlin pattern, and let $G(\mathfrak{T})$ be the product of Gauss sums defined in (41). Let w be the Weyl group element associated to $\mathfrak{T}$ in Proposition 17. Then

$$G(\mathfrak{T}) = \prod_{\alpha \in \Phi_w} g(p^{d_\lambda(\alpha)-1}, p^{d_\lambda(\alpha)}),$$

matching the definition as in (26) where $d_\lambda(\alpha)$ is given by (10).
(ii)

$$H(C_1, \ldots, C_r; m_1 \ldots, m_r) = H_{GT}(C_1, \ldots, C_r; m_1 \ldots, m_r).$$

That is, the Weyl group multiple Dirichlet series is the same as the series defined by the Gelfand–Tsetlin description in the twisted stable case.

Proof. Since both coefficients are obtained from their prime-power parts by means of twisted multiplicativity, part (i) implies part (ii).

We turn to the proof of part (i). Since $\mathfrak{T}$ is stable, we have $s_{ij} = 0$ if $\mathfrak{T}$ is right-leaning at (i,j). Thus

$$G(\mathfrak{T}) = \prod_{(i,j)\text{left-leaning}} g(p^{s_{ij}-1}, p^{s_{ij}}),$$

where the product is over the left-leaning entries of the Gelfand–Tsetlin pattern corresponding to w whose top row is (44).

It suffices to check that the set of s_{ij} at left-leaning entries in the Gelfand–Tsetlin pattern corresponding to w coincides with the set of $d_\lambda(\alpha)$ as α runs over Φ_w. In fact we shall show a slightly sharper statement, namely that the left-leaning entries in row i correspond exactly to a certain set of roots in Φ_w.

To give this more precisely, we require some notation. Recall that we have identified the roots of A_r with the vectors $\mathrm{e}_i - \mathrm{e}_j$, $1 \leq i \neq j \leq r+1$. The action of a permutation $w \in S_{r+1}$ on the corresponding vectors then becomes:

$$w(\mathrm{e}_i - \mathrm{e}_j) = \mathrm{e}_{w(i)} - \mathrm{e}_{w(j)}.$$

Fix w. Observe that (i,j) is an i-inversion for w^{-1} (that is, $i < j$ but $w^{-1}(j) < w^{-1}(i)$) if and only if the root

$$\alpha_{i,j,w} := \mathrm{e}_{w^{-1}(j)} - \mathrm{e}_{w^{-1}(i)}$$

is in Φ_w. Indeed, $\alpha_{i,j,w}$ is positive if and only if $w^{-1}(j) < w^{-1}(i)$, and $w(\alpha_{i,j,w}) = \mathrm{e}_j - \mathrm{e}_i$, which is negative if and only if $j > i$. We will compute the contribution from the set of $\alpha_{i,j,w}$ for each fixed i.

First, we compute $d_\lambda(\alpha_{i,j,w})$. We have

$$\rho = \frac{1}{2} \sum_{m=1}^{r+1} (r+2-2m)\mathrm{e}_m.$$

Also, since $\alpha_i = \mathrm{e}_i - \mathrm{e}_{i+1}$, we have

$$\alpha_{i,j,w} = \sum_{k=w^{-1}(j)}^{w^{-1}(i)-1} \alpha_k.$$

Recall that $\lambda = \sum_{i=1}^{r} l_i \varepsilon_i$, where $\{\varepsilon_i\}$ are the fundamental dominant weights. Since $\langle \alpha_{i,j,w}, \alpha_{i,j,w} \rangle = 2$, we find that

$$d_\lambda(\alpha_{i,j,w}) = 2\frac{\langle \rho + \lambda, \alpha_{i,j,w} \rangle}{\langle \alpha_{i,j,w}, \alpha_{i,j,w} \rangle} = w^{-1}(i) - w^{-1}(j) + \sum_{k=w^{-1}(j)}^{w^{-1}(i)-1} l_k.$$

Now we consider the set of $d_\lambda(\alpha_{i,j,w})$ as j varies over the numbers such that (i,j) is a i-inversion for w^{-1}. We see that $w^{-1}(j)$ runs through the set

$$\{1, \ldots, w^{-1}(i) - 1\} - \{w^{-1}(1), \ldots, w^{-1}(i-1)\},$$

where as usual if X and Y are sets then $X - Y = \{x \in X | x \notin Y\}$. Let

$$D_i = \{d_\lambda(\alpha_{i,j,w}) \mid (i,j) \text{ is an } i\text{-inversion for } w^{-1}\}.$$

Then we obtain the following value for the set D_i:

$$\begin{aligned} D_i = \{&1 + l_{w^{-1}(i)-1}, 2 + l_{w^{-1}(i)-2} + l_{w^{-1}(i)-1}, \ldots, w^{-1}(i) - 1 + l_1 + \cdots + l_{w^{-1}(i)-1}\} \\ &- \{w^{-1}(i) - w^{-1}(j) + l_{w^{-1}(j)} + \cdots + l_{w^{-1}(i)-1} \mid j < i \text{ and } w^{-1}(j) < w^{-1}(i)\}. \end{aligned}$$

Now we turn to the Gauss sums obtained from the Gelfand–Tsetlin pattern. Suppose that there are b_i i-inversions for w^{-1}. By Proposition 19 the first b_i

entries of the i-th row are the left-leaning entries, and the nontrivial Gauss sums in the i-th row come from the quantities s_{ij}, $i \leqslant j \leqslant i + b_i - 1$. Recall that every entry in the stable strict Gelfand–Tsetlin pattern is either left-leaning or right-leaning. We thus have $a_{ij} = a_{i-1,j-1}$ for $i \leqslant j \leqslant i + b_i - 1$ and $a_{ij} = a_{i-1,j}$ for $j \geqslant i + b_i$. The sum for s_{ij} telescopes:

$$\begin{aligned} s_{ij} &= (a_{ij} - a_{i-1,j}) + (a_{i,j+1} - a_{i-1,j+1}) + \cdots + (a_{ir} - a_{i-1,r}) \\ &= (a_{i-1,j-1} - a_{i-1,j}) + (a_{i-1,j} - a_{i-1,j+1}) + \cdots \\ &\quad + (a_{i-1,b_i+i-2} - a_{i-1,b_i+i-1}) + 0 + \cdots + 0 \\ &= a_{i-1,j-1} - a_{i-1,b_i+i-1}. \end{aligned}$$

By Lemma 18, we have

$$a_{i-1,i+b_i-1} = r + 1 - w^{-1}(i) + l_{w^{-1}(i)} + \cdots + l_r.$$

To compute the s_{ij} as j varies, we must subtract this quantity from $a_{i-1,i-1+k}$ for each k, $0 \leqslant k \leqslant b_i - 1$. So we must compute the quantities $a_{i-1,i-1+k}$. Recall that the 0-th row of the Gelfand–Tsetlin pattern is $\{l(r), l(r-1), \ldots, l(0)\}$. By Lemma 18 again, the entries of the $(i-1)$-th row of the Gelfand–Tsetlin pattern are given by

$$\{l(r), \ldots, l(0)\} - \{l(r + 1 - w^{-1}(m)) \mid 1 \leqslant m \leqslant i - 1\}. \tag{51}$$

We need to specify the b_i entries with largest argument in the set (51); these are the elements from which we will subtract the term $l(r + 1 - w^{-1}(i))$. We have

$$b_i = |\{j > i \mid w^{-1}(j) < w^{-1}(i)\}| = w^{-1}(i) - 1 - |\{j < i \mid w^{-1}(j) < w^{-1}(i)\}|.$$

Let $h_1, \ldots, h_{b_i}$ be the integers in the interval $[1, w^{-1}(i) - 1]$ that are not of the form $w^{-1}(j)$ with some j, $j < i$. These are the only integers h in the interval $[1, w^{-1}(i) - 1]$ such that $l(r + 1 - h)$ is not removed from the $(i-1)$-th row of the Gelfand–Tsetlin pattern, by (51). Hence the only terms of the form $l(r + 1 - k)$ with $1 \leqslant k < w^{-1}(i)$ that are in the Gelfand–Tsetlin pattern are exactly the numbers of the form $l(r + 1 - h_m)$, $1 \leqslant m \leqslant b_i$. Since these are visibly the b_i entries with largest argument, we have determined the entries $a_{i-1,i-1+k}$, $0 \leqslant k \leqslant b_i - 1$. We have

$$l(r + 1 - h_m) - l(r + 1 - w^{-1}(i)) = w^{-1}(i) - h_m + l_{h_m} + \cdots + l_{w^{-1}(i)-1}.$$

As m varies from 1 to b_i, these quantities give exactly the set D_i that we obtained above from the formula for the coefficients (26).

This completes the proof of Theorem 20. □

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A Topological Model for Some Summand of the Eisenstein Cohomology of Congruence Subgroups

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Summary. We construct a topological model for certain Eisenstein cohomology of congruence subgroups.

1 Introduction

Let $\mathcal{G}$ be a semisimple algebraic group over $\mathbb{Q}$, let $\mathcal{G}(\mathbb{Q})$ and $\mathcal{G}(\mathbb{A})$ be its rational and adelic groups, and let $\boldsymbol{K} \subset \mathcal{G}(\mathbb{A})$ be a good maximal compact subgroup. Let $\boldsymbol{K} = \boldsymbol{K}_f \boldsymbol{K}_\infty$ with $\boldsymbol{K}_\infty \subset \mathcal{G}(\mathbb{R})$ and $\boldsymbol{K}_f \subset \mathcal{G}(\mathbb{A}_f)$, where $\mathcal{G}(\mathbb{A}_f)$ is the finite adelic group and $\mathcal{G}(\mathbb{R})$ is the group of real points. By our assumption on $\mathcal{G}$, we know that $\mathcal{G}(\mathbb{R})$ and $\boldsymbol{K}_\infty$ are connected Lie groups (cf. Proposition 2.1 below). Then the cohomology of the congruence subgroup $\Gamma = \mathcal{G}(\mathbb{Q}) \cap \boldsymbol{K}_f$ can be computed by

$$H^*(\Gamma, \mathbb{C}) = H^*\left(\mathcal{G}(\mathbb{Q})\backslash\mathcal{G}(\mathbb{A})/\boldsymbol{K}_\infty, \mathbb{C}\right)^{\boldsymbol{K}_f}, \tag{1.1}$$

where the superscript $\boldsymbol{K}_f$ stands for the subspace of $\boldsymbol{K}_f$-invariants in the $\mathcal{G}(\mathbb{A}_f)$-module

$$H^*\left(\mathcal{G}(\mathbb{Q})\backslash\mathcal{G}(\mathbb{A})/\boldsymbol{K}_\infty, \mathbb{C}\right) := \operatorname*{colim}_{\boldsymbol{K}^f} H^*\left(\mathcal{G}(\mathbb{Q})\backslash\mathcal{G}(\mathbb{A})/\boldsymbol{K}^f\boldsymbol{K}_\infty, \mathbb{C}\right). \tag{1.2}$$

The inductive limit is over all open subgroups $\boldsymbol{K}^f \subseteq \boldsymbol{K}_f$. It is clear from definition 1.1 that the Hecke algebra $\mathfrak{H} = C_c^\infty\left(\boldsymbol{K}_f\backslash\mathcal{G}(\mathbb{A}_f)/\boldsymbol{K}_f\right)$ of compactly supported $\boldsymbol{K}_f$-biinvariant functions on $\mathcal{G}(\mathbb{A}_f)$ acts on $H^*(\Gamma, \mathbb{C})$. Let

$$\mathcal{I} := \left\{ f \in \mathfrak{H} = C_c^\infty(\boldsymbol{K}_f\backslash\mathcal{G}(\mathbb{A}_f)/\boldsymbol{K}_f) \;\middle|\; \int_{\mathcal{G}(\mathbb{A}_f)} f(g)\,dg \right\} = 0$$

be the ideal of elements of $\mathfrak{H}$ which act trivially on the constant representation. Since $H^*(\Gamma, \mathbb{C})$ is a finite dimensional vector space, any element of $H^*(\Gamma, \mathbb{C})$ is annihilated by a finite power of an ideal of finite codimension in $\mathfrak{H}$. Therefore, the subspace

$$H^*(\Gamma,\mathbb{C})_{\mathcal{I}} = \{x \in H^*(\Gamma,\mathbb{C})\,|\mathcal{I}^n x = \{0\} \text{ for some } n > 0\}$$

is a direct summand of $H^*(\Gamma,\mathbb{C})$ which, among other elements, contains the constant cohomology class in dimension zero. One of the aims of this article is to study the space $H^*(\Gamma,\mathbb{C})_{\mathcal{I}}$.

Our main result gives a topological model for $H^*(\Gamma,\mathbb{C})_{\mathcal{I}}$. We first recall the topological model for the cohomology of the constant representation of $\mathcal{G}(\mathbb{R})$, which maps to $H^*(\Gamma,\mathbb{C})_{\mathcal{I}}$. Let $I^*_{\mathcal{G}(\mathbb{R}),\boldsymbol{K}_\infty}$ be the algebra of $\mathcal{G}(\mathbb{R})$-invariant differential forms on the symmetric space $\mathcal{G}(\mathbb{R})/\boldsymbol{K}_\infty$. Such forms are closed and give rise to $\mathcal{G}(\mathbb{A}_f)$-invariant elements in $H^*\left(\mathcal{G}(\mathbb{Q})\backslash\mathcal{G}(\mathbb{A})/\boldsymbol{K}_\infty,\mathbb{C}\right)$. We get a map of graded vector spaces

$$I^*_{\mathcal{G}(\mathbb{R}),\boldsymbol{K}_\infty} \to H^*(\Gamma,\mathbb{C})_{\mathcal{I}}.$$

Furthermore, $H^*(\Gamma,\mathbb{C})_{\mathcal{I}}$ is a $I^*_{\mathcal{G}(\mathbb{R}),\boldsymbol{K}_\infty}$-module since multiplication by $\mathcal{G}(\mathbb{A}_f)$-invariant cohomology classes, unlike the rest of the multiplicative structure, commutes with the action of the Hecke algebra. Let $\mathcal{G}^{(c)}(\mathbb{R}) \subset \mathcal{G}(\mathbb{C})$ be a compact form of $\mathcal{G}(\mathbb{R})$ such that $\boldsymbol{K}_\infty \subset \mathcal{G}^{(c)}(\mathbb{R})$. Then the homogeneous space $\boldsymbol{X}^{(c)}_{\mathcal{G}} :\ = \mathcal{G}^{(c)}(\mathbb{R})/\boldsymbol{K}_\infty$ is the compact dual of $\mathcal{G}(\mathbb{R})/\boldsymbol{K}_\infty$. The complexified tangent spaces at the origins of $\boldsymbol{X}^{(c)}_{\mathcal{G}}$ and of $\mathcal{G}(\mathbb{R})/\boldsymbol{K}_\infty$ can be identified, and one gets an identification of $I^*_{\mathcal{G}(\mathbb{R}),\boldsymbol{K}_\infty}$ with the space of $\mathcal{G}^{(c)}(\mathbb{R})$-invariant forms on $\boldsymbol{X}^{(c)}_{\mathcal{G}}$. The space of $\mathcal{G}^{(c)}(\mathbb{R})$-invariant forms on $\boldsymbol{X}^{(c)}_{\mathcal{G}}$ is equal to the space of harmonic forms (with respect to a $\mathcal{G}^{(c)}(\mathbb{R})$-invariant metric) on $\boldsymbol{X}^{(c)}_{\mathcal{G}}/\boldsymbol{K}_\infty$, hence it is isomorphic to $H^*(\boldsymbol{X}^{(c)}_{\mathcal{G}},\mathbb{C})$. We obtain a multiplicative isomorphism between $I^*_{\mathcal{G}(\mathbb{R}),\boldsymbol{K}_\infty}$ and $H^*(\boldsymbol{X}^{(c)}_{\mathcal{G}},\mathbb{C})^{\pi_0(\boldsymbol{K}_\infty)}$.

Our topological model for $H^*(\Gamma,\mathbb{C})_{\mathcal{I}}$ consists of a canonical isomorphism of $I^*_{\mathcal{G}(\mathbb{R}),\boldsymbol{K}_\infty} \cong H^*(\boldsymbol{X}^{(c)}_{\mathcal{G}},\mathbb{C})$-modules from $H^*(\Gamma,\mathbb{C})_{\mathcal{I}}$ onto the invariants of a certain group in $H^*(\boldsymbol{U}_{\mathcal{G}},\mathbb{C})$, where $\boldsymbol{U}_{\mathcal{G}} \subset \boldsymbol{X}^{(c)}_{\mathcal{G}}$ is a certain open subset. To give the definition of $\boldsymbol{U}_{\mathcal{G}}$, we first have to introduce some new notation. Let $\mathcal{P}_o$ be a minimal $\mathbb{Q}$-rational parabolic subgroup of $\mathcal{G}$. We consider standard parabolic subgroups $\mathcal{P} \supseteq \mathcal{P}_o$. Let $\mathcal{N}_{\mathcal{P}} \subset \mathcal{P}$ be the radical of $\mathcal{P}$ and let $\mathcal{L}_{\mathcal{P}} = \mathcal{P}/\mathcal{N}_{\mathcal{P}}$. Let

$$\mathcal{M}_{\mathcal{P}} :\ = \Big(\bigcap_{\chi \in X^*(\mathcal{L}_{\mathcal{P}})} \ker(\chi)\Big)^o \tag{1.3}$$

be the connected component of the intersection of the kernels of all $\mathbb{Q}$-rational characters of $\mathcal{L}_{\mathcal{P}}$. To ensure that our constructions do not depend on such a choice, we will never choose a $\mathbb{Q}$-rational section $\mathcal{L}_{\mathcal{P}} \to \mathcal{P}$ of the canonical projection $\mathcal{P} \to \mathcal{L}_{\mathcal{P}}$. We will, however, use the fact that the projection $\mathcal{P} \cap \theta(\mathcal{P}) \to \mathcal{L}_{\mathcal{P}}$, where θ is the Cartan involution defined by $\boldsymbol{K}_\infty$, is an isomorphism of algebraic groups over $\mathbb{R}$. This identifies $\mathcal{L}_{\mathcal{P}}(\mathbb{R})$ and $\mathcal{L}_{\mathcal{P}}(\mathbb{C})$ with subgroups of $\mathcal{G}(\mathbb{R})$ and $\mathcal{G}(\mathbb{C})$. Using this identification, the compact form of $\mathcal{M}_{\mathcal{P}}$ becomes

$$\mathcal{M}_{\mathcal{P}}^{(c)}(\mathbb{R}) = \mathcal{M}_{\mathcal{P}}(\mathbb{C}) \cap \mathcal{G}^{(c)}(\mathbb{R}),$$

a subgroup of the compact form of $\mathcal{G}$, and the compact dual of the symmetric space defined by $\mathcal{M}_{\mathcal{P}}$ becomes

$$\boldsymbol{X}_{\mathcal{M}_{\mathcal{P}}}^{(c)} = \mathcal{M}_{\mathcal{P}}^{(c)}(\mathbb{R})/(K_\infty \cap \mathcal{M}_{\mathcal{P}}(\mathbb{R})) \subset \boldsymbol{X}_{\mathcal{G}}^{(c)},$$

a subset of the compact dual of the symmetric space defined by $\mathcal{G}$. We put

$$\boldsymbol{U}_{\mathcal{G}} \colon = \boldsymbol{X}_{\mathcal{G}}^{(c)} - \bigcup_{\mathcal{P} \supseteq \mathcal{P}_o} \boldsymbol{X}_{\mathcal{M}_{\mathcal{P}}}^{(c)}. \tag{1.4}$$

The group $\boldsymbol{K}_\infty \cap \mathcal{P}_o(\mathbb{R})$ acts on $\boldsymbol{X}_{\mathcal{G}}^{(c)}$ by left translations and leaves $\boldsymbol{U}_{\mathcal{G}}$ invariant. The action of $\boldsymbol{K}_\infty \cap \mathcal{P}_o(\mathbb{R})$ on the cohomology of $\boldsymbol{X}_{\mathcal{G}}^{(c)}$ is trivial, the action on the cohomology of $\boldsymbol{U}_{\mathcal{G}}$ factorizes over the finite group of connected components $\pi_0\big(\boldsymbol{K}_\infty \cap \mathcal{P}_o(\mathbb{R})\big)$. With these definitions, we can formulate our main result about $H^*(\Gamma, \mathbb{C})_{\mathcal{I}}$.

Theorem 1.1. *There is a canonical isomorphism of* $I^*_{\mathcal{G}(\mathbb{R}),\boldsymbol{K}_\infty} \cong H^*(\boldsymbol{X}_{\mathcal{G}}^{(c)}, \mathbb{C})$*-modules*

$$H^*(\Gamma, \mathbb{C})_{\mathcal{I}} \cong H^*(\boldsymbol{U}_{\mathcal{G}}, \mathbb{C})^{\boldsymbol{K}_\infty \cap \mathcal{P}_o(\mathbb{R})}. \tag{1.5}$$

Furthermore, elements of $H^*(\Gamma, \mathbb{C})_{\mathcal{I}}$*, which by definition are annihilated by some power of* $\mathcal{I} \subset \mathfrak{H}$*, are already annihilated by* $\mathcal{I}$ *itself.*

The map $I^*_{\mathcal{G}(\mathbb{R}),\boldsymbol{K}_\infty} \to H^*(\Gamma, \mathbb{C})$ was first studied by Borel [2], who proved that it is an isomorphism in low dimension. Since $H^*(\Gamma, \mathbb{C})_{\mathcal{I}}$ is a direct summand of $H^*(\Gamma, \mathbb{C})$, the question of noninjectivity of Borel's map (which was studied by Speh [22]), can be understood in terms of restriction of cohomology classes from $\boldsymbol{X}_{\mathcal{G}}^{(c)}$ to $\boldsymbol{U}_{\mathcal{G}}$. Our interest in this particular summand was, however, motivated by the fact that it is an important model case for the effects produced by the singularities of Eisenstein series when one studies the cohomology of congruence subgroups in terms of automorphic forms. Our method of studying $H^*(\Gamma, \mathbb{C})_{\mathcal{I}}$ uses the results of [9]. It consists of expressing $H^*(\Gamma, \mathbb{C})_{\mathcal{I}}$ as the $(\mathfrak{g}, K)$-cohomology of a direct summand of the space of automorphic forms and of representing this space in terms of Eisenstein series. The Eisenstein series which are of interest are the Eisenstein series starting from the constant functions on the Levi components of standard parabolic subgroups, evaluated at one half the sum of the positive roots. There are many singular hyperplanes which go through this parameter, and the iterated residue of the Eisenstein series is the constant function on $\mathcal{G}(\mathbb{A})$. The contributions from the Eisenstein series starting from a given parabolic subgroup is therefore no direct summand of the space of automorphic forms, but only a quotient of a suitable filtration on the space of automorphic forms. The problem of understanding these extensions was the main motivation for writing this paper. For GL_2 over algebraic number fields, the summand of the cohomology considered in this paper, has been computed by Harder [13, Theorem 4.2.2.]. There are probably more explicit calculations for rank one cases and also some for rank two cases, for instance in [20]. These authors do not use topological models to describe the Eisenstein cohomology, they arrive at explicit formulas.

We can more generally study the $\mathcal{G}(\mathbb{A}_f)$-module of all elements x in the cohomology $H^*\left(\mathcal{G}(\mathbb{Q})\backslash\mathcal{G}(\mathbb{A})/\boldsymbol{K}_\infty,\mathbb{C}\right)$ which at all but the finitely many ramified places are annihilated by some power of $\mathcal{I}$. Again it turns out that the first power is sufficient. Let $H^*(\mathcal{G})_\mathcal{I}$ be the space of cohomology classes x with that property. Then $H^*(\mathcal{G})_\mathcal{I}$ can be identified with the $\boldsymbol{K}_\infty \cap \mathcal{P}_o(\mathbb{R})$-invariants in the hypercohomology of a complex of sheaves with $\mathcal{G}(\mathbb{A}_f)$-action on $\boldsymbol{X}_\mathcal{G}^{(c)}$. It turns out that the hypercohomology spectral sequence for this complex degenerates, and that the limit filtration can be described in terms of the $\mathcal{G}(\mathbb{A}_f)$-action. However, Hilbert modular forms and SL_3 over imaginary quadratic fields provide easy examples that the limit filtration will usually not split in the category of $\mathcal{G}(\mathbb{A}_f)$-modules. To get a complete picture of $H^*(\mathcal{G})_\mathcal{I}$ as a $\mathcal{G}(\mathbb{A}_f)$-module, one may be forced to carry out the laborious work of explicit calculations for the various families of algebraic groups. As an example, we carry out explicit calculations for SL_n over imaginary quadratic fields. This example shows that while explicit calculations for the various series of classical groups should be possible, the topological model provides a much more vivid picture of the cohomology.

By the work of Moeglin and Waldspurger [17], the residual spectrum of GL_n over a number field is now completely understood. The structure of the residues is quite similar to the case investigated in this paper. Therefore, there is some hope that our methods can be used to completely understand the Eisenstein cohomology of GL_n in terms of the cuspidal cohomological representations. Compared with this paper, one has to expect two difficulties. Firstly, there is the possibility of "overlapping Speh segments". In this case, the structure of the Eisenstein cohomology may depend on whether some automorphic L-function vanishes at the center of the functional equation. This effect was first found by Harder [14, §III] in the case of GL_3 over imaginary fields. As a second complication, the Borel–Serre–Solomon–Tits Theorem 4.2 in this paper will not suffice. One needs a Solomon–Tits type theorem with twisted coefficients, which investigates the cohomology of a complex formed by normalised intertwining operators. I hope that the methods of this paper are flexible enough to extend to this new situation.

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2 Notations

We will study connected reductive linear algebraic groups $\mathcal{G}$ over $\mathbb{Q}$. Let $\boldsymbol{K} = \boldsymbol{K}_f \boldsymbol{K}_\infty$ be a good maximal compact subgroup of $\mathcal{G}(\mathbb{A})$, decomposed into its finite adelic factor $\boldsymbol{K}_f$ and its real factor $\boldsymbol{K}_\infty$. Let θ be the Cartan involution with respect to $\boldsymbol{K}_\infty$, and let $\boldsymbol{K}_\infty^o$ be the connected component of $\boldsymbol{K}_\infty$. We denote by $\mathcal{P}_o$ a fixed minimal $\mathbb{Q}$-rational parabolic subgroup of $\mathcal{G}$. Unless otherwise specified, parabolic subgroups $\mathcal{P}$ will be assumed to be defined over $\mathbb{Q}$ and to be standard with respect to $\mathcal{P}_o$. Let $\mathcal{N}_\mathcal{P}$ be the radical of $\mathcal{P}$, and let $\mathcal{L}_\mathcal{P} = \mathcal{P}/\mathcal{N}_\mathcal{P}$ be the Levi component. Unless $\mathcal{P} = \mathcal{G}$, we will not think of $\mathcal{L}_\mathcal{P}$ as a subgroup of $\mathcal{P}$. We will, however, identify $\mathcal{L} \times_{\mathrm{Spec}(\mathbb{Q})} \mathrm{Spec}(\mathbb{R})$ with the $\mathbb{R}$-rational algebraic subgroup $\mathcal{P} \cap \theta(\mathcal{P})$ of $\mathcal{L}_\mathcal{P}$. Let $\mathcal{A}_\mathcal{P}$ be a maximal $\mathbb{Q}$-split torus in the center of $\mathcal{L}_\mathcal{P}$, and let $\mathcal{M}_\mathcal{P}$ be defined by (1.3), such that $\mathcal{L}_\mathcal{P} = \mathcal{A}_\mathcal{P}\mathcal{M}_\mathcal{P}$ is an isogeny. In the case $\mathcal{P} = \mathcal{P}_o$, we will write $\mathcal{M}_o$, $\mathcal{A}_o$, and $\mathcal{N}_o$ instead of $\mathcal{M}_{\mathcal{P}_o}$, $\mathcal{A}_{\mathcal{P}_o}$, and $\mathcal{N}_{\mathcal{P}_o}$. In the case $\mathcal{P} = \mathcal{G}$, $\mathcal{A}_\mathcal{G}$ is a maximal $\mathbb{Q}$-split torus in the center of $\mathcal{G}$, and $\mathcal{M}_\mathcal{G}$ is generated by the derived group of $\mathcal{G}$ and the $\mathbb{Q}$-anisotropic part of the center of $\mathcal{G}$.

Let $\mathcal{G}(\mathbb{A})$ be the adelic group of $\mathcal{G}$. If S is a subset of the set of valuations of $\mathbb{Q}$, let $\mathcal{G}(\mathbb{A}_S)$ be the restricted product over all places $v \in S$ of the groups $\mathcal{G}(\mathbb{Q}_v)$. In the special case where S is the set of finite primes, this is the finite adelic group $\mathcal{G}(\mathbb{A}_f)$. Let $\boldsymbol{K}_S = \boldsymbol{K} \cap \mathcal{G}(\mathbb{A}_S)$. For a parabolic subgroup $\mathcal{P}$, let $\mathcal{A}_\mathcal{P}(\mathbb{R})^+$ be the connected component of the group of real points $\mathcal{A}_\mathcal{P}(\mathbb{R})$. In the special case $\mathcal{P} = \mathcal{G}$, this is the connected component of the group of real points of a maximal $\mathbb{Q}$-split torus in the center of $\mathcal{G}$.

Let $\mathfrak{g}$ be the Lie algebra of $\mathcal{G}(\mathbb{R})$, $\mathfrak{U}(\mathfrak{g})$ its universal enveloping algebra, and $\mathfrak{Z}(\mathfrak{g})$ the center of $\mathfrak{U}(\mathfrak{g})$. Similar notations will be used for the Lie algebras of other groups.

Let $\mathfrak{a}_\mathcal{P}$ be the Lie algebra of $\mathcal{A}_\mathcal{P}(\mathbb{R})$. We will write $\mathfrak{a}_o$ for $\mathfrak{a}_{\mathcal{P}_o}$. If $\mathcal{P} \subseteq \mathcal{Q}$, then it is possible to choose a section $i_\mathcal{Q}\colon \mathcal{L}_\mathcal{Q} \to \mathcal{Q}$ of the projection $\mathcal{Q} \to \mathcal{L}_\mathcal{Q}$. Then $i_\mathcal{Q}\big(\mathrm{pr}_{\mathcal{Q}\to\mathcal{L}_\mathcal{Q}}(\mathcal{P})\big) \subset \mathcal{P}$. We define an embedding $\mathfrak{a}_\mathcal{Q} \to \mathfrak{a}_\mathcal{P}$ as the restriction to $\mathfrak{a}_\mathcal{Q}$ of the differential of the map

$$\mathrm{pr}_{\mathcal{P}\to\mathcal{L}_\mathcal{P}} i_\mathcal{Q}.$$

This embedding is independent of the choice of $i_\mathcal{Q}$. The dual space $\check{\mathfrak{a}}_\mathcal{P}$ of $\mathfrak{a}_\mathcal{P}$ can be identified with the real vector space $X^*(\mathcal{P}) \otimes_\mathbb{Z} \mathbb{R}$ generated by the group of $\mathbb{Q}$-rational characters of $\mathcal{P}$. The same identification can be made for $\mathcal{Q}$. Then restriction of characters from $\mathcal{Q}$ to $\mathcal{P}$ defines an embedding $\check{\mathfrak{a}}_\mathcal{Q} \to$

$\check{\mathfrak{a}}_{\mathcal{P}}$. The embeddings $\mathfrak{a}_{\mathcal{Q}} \to \mathfrak{a}_{\mathcal{P}}$ and $\check{\mathfrak{a}}_{\mathcal{Q}} \to \check{\mathfrak{a}}_{\mathcal{P}}$ define canonical direct sum decompositions $\mathfrak{a}_{\mathcal{P}} = \mathfrak{a}_{\mathcal{Q}} \oplus \mathfrak{a}_{\mathcal{P}}^{\mathcal{Q}}$ and $\check{\mathfrak{a}}_{\mathcal{P}} = \check{\mathfrak{a}}_{\mathcal{Q}} \oplus \check{\mathfrak{a}}_{\mathcal{P}}^{\mathcal{Q}}$.

Let $\Delta_o \subset \check{\mathfrak{a}}_o^{\mathcal{G}}$ be the set of simple positive (with respect to $\mathcal{P}_o$) roots of $\mathcal{A}_o$. The subset $\Delta_o^{\mathcal{P}}$ of simple positive roots which occur in the Lie algebra of $\mathcal{M}_{\mathcal{P}}$ is contained in $\check{\mathfrak{a}}_o^{\mathcal{P}}$. Of course, both definitions require the choice of sections $\mathcal{L}_o \to \mathcal{L}_{\mathcal{P}} \to \mathcal{P}$, but the result does not depend on such a choice. Let $\Delta_{\mathcal{P}}$ be the projection of $\Delta_o - \Delta_o^{\mathcal{P}}$ to $\check{\mathfrak{a}}_{\mathcal{P}}$, and let $\Delta_{\mathcal{P}}^{\mathcal{Q}}$ for $\mathcal{P} \subseteq \mathcal{Q}$ be the projection of $\Delta_o^{\mathcal{Q}} - \Delta_o^{\mathcal{P}}$ to $\check{\mathfrak{a}}_{\mathcal{P}}^{\mathcal{Q}}$. Let $\rho_o \in \check{\mathfrak{a}}_o$ be one half the sum of the positive roots of $\mathcal{A}_o$, and let $\rho_{\mathcal{P}}$ and $\rho_{\mathcal{P}}^{\mathcal{Q}}$ be the projections of ρ_o to $\mathfrak{a}_{\mathcal{P}}$ and to $\mathfrak{a}_{\mathcal{P}}^{\mathcal{Q}}$.

Our notion of a $(\mathfrak{g}, K)$-module is the same as in [26, §6.1]. A $\mathcal{G}(\mathbb{A}_f)$-module is a vector space on which $\mathcal{G}(\mathbb{A}_f)$ acts with open stabilizers. If $\mathbb{K}$ is a field, let $C_c^\infty(\mathcal{G}(\mathbb{A}_f), \mathbb{K})$ be the $\mathcal{G}(\mathbb{A}_f)$-module of compactly supported locally constant $\mathbb{K}$-valued functions on $\mathcal{G}(\mathbb{A}_f)$. If no field is given, it is assumed that $\mathbb{K} = \mathbb{C}$. A similar notation is used for quotients of the adelic group. For quotients of the full adelic group like $C^\infty(\mathcal{P}(\mathbb{A})\backslash\mathcal{G}(\mathbb{A}))$ or similar quotients of partial adelic groups which contain $\mathcal{G}(\mathbb{R})$, we adopt the conditions that C^∞-functions have to be locally constant with respect to the finite adelic part and $\boldsymbol{K}_\infty$-finite and infinitely often differentiable with respect to $\mathcal{G}(\mathbb{R})$.

2.1 Connected components of real groups

Let us recall the following fact:

Proposition 2.1. *Let $\mathcal{G}$ be a reductive connected algebraic group over $\mathbb{R}$ and let $\boldsymbol{K}_\infty$ be a maximal compact subgroup of $\mathcal{G}(\mathbb{R})$.*

1. *Then $\pi_0(\boldsymbol{K}_\infty) \cong \pi_0(\mathcal{G}(\mathbb{R}))$.*
2. *If $\mathcal{R} \subset \mathcal{Q}$ are parabolic subgroups defined over $\mathbb{R}$, then the map*

$$\pi_0(\mathcal{R}(\mathbb{R}) \cap \boldsymbol{K}_\infty) \to \pi_0(\mathcal{Q}(\mathbb{R}) \cap \boldsymbol{K}_\infty)$$

   is surjective.
3. *If $\mathcal{G}$ is $\mathbb{R}$-anisotropic or if it is semisimple and simply connected, then $\mathcal{G}(\mathbb{R})$ is connected.*

Proof. The first two assertions are consequences of the Iwasawa decomposition $\mathcal{G}(\mathbb{R}) \cong \mathcal{P}(\mathbb{R})^o \times \boldsymbol{K}_\infty$, where $\mathcal{P}$ is a minimal $\mathbb{R}$-parabolic subgroup (cf. [23, Proposition 5.15]). The third fact is established in [6, Corollaire 4.7] for semisimple simply connected groups and in [5, Corollaire 14.5] for anisotropic groups. □

3 Formulation of the main results

Let $H^*(\mathcal{G})$ be the inductive limit

$$H^*(\mathcal{G}) := \operatorname*{colim}_{\boldsymbol{K}^f} H^*\left(\mathcal{G}(\mathbb{Q})\mathcal{A}_{\mathcal{G}}(\mathbb{R})^+\backslash\mathcal{G}(\mathbb{A})/\boldsymbol{K}^f\boldsymbol{K}_\infty^o, \mathbb{C}\right) \tag{3.1}$$

over all sufficiently small compact open subgroups $\boldsymbol{K}^f \subset \mathcal{G}(\mathbb{A}_f)$. This is a $\mathcal{G}(\mathbb{A}_f)$-module. Let $H_c^*(\mathcal{G}, \mathbb{C})$ be the same inductive limit over the cohomology with compact support. For any set of finite primes S, the Hecke algebra $\mathfrak{H}_S = C_c^\infty(\boldsymbol{K}_S \backslash \mathcal{G}(\mathbb{A}_S)/\boldsymbol{K}_S)$ of $\boldsymbol{K}_S$-bi-invariant compactly supported functions on $\mathcal{G}(\mathbb{A}_S)$ acts on $H^*(\mathcal{G}, \mathbb{C})$ and $H_c^*(\mathcal{G}, \mathbb{C})$. Let $\mathcal{I}_S$ be the ideal

$$\mathcal{I}_S \colon = \left\{ f \in \mathfrak{H}_S \;\middle|\; \int_{\mathcal{G}(\mathbb{A}_S)} f(g)\, dg = 0 \right\},$$

and let

$$\begin{aligned}
&H^*(\mathcal{G}, \mathbb{C})_{\mathcal{I}} \colon = \\
&\{x \in H^*(\mathcal{G}, \mathbb{C}) \,| \text{for any set } S \text{ of finite primes, } \mathcal{I}_S^m x = \{0\} \text{ for } m \gg 0\} \\
&H_c^*(\mathcal{G}, \mathbb{C})_{\mathcal{I}} \colon = \\
&\{x \in H_c^*(\mathcal{G}, \mathbb{C}) \,| \text{for any set } S \text{ of finite primes, } \mathcal{I}_S^m x = \{0\} \text{ for } m \gg 0\}.
\end{aligned}$$

These are direct summands of $H^*(\mathcal{G}, \mathbb{C})$ and $H_c^*(\mathcal{G}, \mathbb{C})$. Our main result describes them as the space of $\boldsymbol{K}_\infty^o \cap \mathcal{P}_o(\mathbb{R})$-invariants in the hypercohomology of a complex of sheaves of $\mathcal{G}(\mathbb{A}_f)$-modules on the compact dual.

The construction of these complexes of sheaves follows a general pattern, which associates a chain complex to a functor with values in an abelian category on the poset $\mathfrak{P}_{\mathcal{G}}$ of standard parabolic subgroups. Note that $\mathcal{G}$ is a maximal element of $\mathfrak{P}_{\mathcal{G}}$. Let $\prec$ be a total order on Δ_o. We order successors $\mathcal{Q}$ of $\mathcal{P}$ in $\mathfrak{P}_{\mathcal{G}}$ by the order $\prec$ of the unique element of $\Delta_o^{\mathcal{Q}} - \Delta_o^{\mathcal{P}}$ and denote the i-th successor $(0 \le i < \dim \mathfrak{a}_{\mathcal{P}}^{\mathcal{G}})$ of $\mathcal{P}$ by $\mathcal{P}_i$. Let $\boldsymbol{F}^{\mathcal{P}}$ be a contravariant functor on $\mathfrak{P}_{\mathcal{G}}$. For $\mathcal{P} \subseteq \mathcal{Q}$, let

$$\boldsymbol{F}^{\mathcal{P} \subseteq \mathcal{Q}} \colon \boldsymbol{F}^{\mathcal{Q}} \to \boldsymbol{F}^{\mathcal{P}}$$

be the transition map. We define the chain complex $C^*(\boldsymbol{F}^\bullet)$ by

$$C^k(\boldsymbol{F}^\bullet) = \bigoplus_{\substack{\mathcal{P} \in \mathfrak{P} \\ \dim \mathfrak{a}_{\mathcal{P}}^{\mathcal{G}} = k}} \boldsymbol{F}^{\mathcal{P}}$$

with the differential

$$d\left((f_{\mathcal{P}})_{\substack{\mathcal{P} \in \mathfrak{P} \\ \dim \mathfrak{a}_{\mathcal{P}}^{\mathcal{G}} = k}} \right) = \left(\sum_{i=0}^{k} (-1)^i \boldsymbol{F}^{\mathcal{Q} \subset \mathcal{Q}_i}(f_{\mathcal{Q}_i}) \right)_{\substack{\mathcal{Q} \in \mathfrak{P} \\ \dim \mathfrak{a}_{\mathcal{Q}}^{\mathcal{G}} = k+1}} . \tag{3.2}$$

Similarly, let $\boldsymbol{F}_{\mathcal{P}}$ be covariant, with transition maps

$$\boldsymbol{F}_{\mathcal{P} \subseteq \mathcal{Q}} \colon \boldsymbol{F}_{\mathcal{P}} \to \boldsymbol{F}_{\mathcal{Q}}.$$

We order predecessors $\mathcal{Q}$ of $\mathcal{P}$ in $\mathfrak{P}$ according to the order by $\prec$ of the unique element of $\Delta_o^{\mathcal{P}} - \Delta_o^{\mathcal{Q}}$, denote the i-th predecessor $(0 \le i < \dim \mathfrak{a}_o^{\mathcal{P}})$ by ${}_i\mathcal{P}$ and form the chain complex

$$C^k(\boldsymbol{F}_\bullet) = \bigoplus_{\substack{\mathcal{P}\in\mathfrak{P}\\ \dim \mathfrak{a}_o^{\mathcal{P}}=k}} \boldsymbol{F}_{\mathcal{P}}$$

with differential

$$d\left((f_{\mathcal{P}})_{\substack{\mathcal{P}\in\mathfrak{P}\\ \dim \mathfrak{a}_o^{\mathcal{P}}=k}}\right) = \left(\sum_{i=0}^{k}(-1)^i \boldsymbol{F}_{{}_i\mathcal{Q}\subset\mathcal{Q}}(f_{{}_i\mathcal{Q}})\right)_{\substack{\mathcal{Q}\in\mathfrak{P}\\ \dim \mathfrak{a}_o^{\mathcal{Q}}=k+1}}. \tag{3.3}$$

We apply similar conventions to functors of several variables. For instance, if $\boldsymbol{F}_{\mathcal{P}}^{\mathcal{Q}}$ is covariant with respect to $\mathcal{P}$ and contravariant with respect to $\mathcal{Q}$, then we have the following chain complexes:

- For fixed $\mathcal{P}$, the chain complex $C^*(\boldsymbol{F}_{\mathcal{P}}^\bullet)$ obtained by applying construction 3.2 to the contravariant variable.
- For fixed $\mathcal{Q}$, the chain complex $C^*(\boldsymbol{F}_\bullet^{\mathcal{Q}})$ obtained by applying construction 3.3 to the covariant variable.
- The chain complex $C^*(\boldsymbol{F}_\bullet^\bullet)$, the total complex of the double complex, obtained by applying (3.2) to the contravariant variable and (3.3) to the covariant variable.

Of course, all these complexes depend on the choice of $\prec$. However, they do so only up to unique isomorphism. For instance, let $C^*(\boldsymbol{F}^{\mathcal{P}})_\prec$ be formed with respect to $\prec$ and let $C^*(\boldsymbol{F}^{\mathcal{P}})_{\tilde{\prec}}$ be formed with respect to $\tilde{\prec}$. Then we have the isomorphism of complexes

$$\begin{aligned} C^*(\boldsymbol{F}^\bullet)_\prec &\to C^*(\boldsymbol{F}^{\mathcal{P}})_{\tilde{\prec}} \\ (f_{\mathcal{P}})_{\substack{\mathcal{P}\in\mathfrak{P}\\ \dim \mathfrak{a}_{\mathcal{P}}^{\mathcal{G}}=k}} &\to (\varepsilon_{\mathcal{P}} f_{\mathcal{P}})_{\substack{\mathcal{P}\in\mathfrak{P}\\ \dim \mathfrak{a}_{\mathcal{P}}^{\mathcal{G}}=k}}, \end{aligned}$$

where $\varepsilon_{\mathcal{P}}$ is the signature of the permutation of $\Delta_o - \Delta_o^{\mathcal{P}}$ which identifies the total orders $\prec$ and $\tilde{\prec}$ of $\Delta_o - \Delta_o^{\mathcal{P}}$. We will therefore suppress the $\prec$-dependence of $C^*(\boldsymbol{F}^\bullet)$ in our notations. The same applies to $C^*(\boldsymbol{F}_\bullet)$ and the constructions for bifunctors. We will also apply these constructions if $\boldsymbol{F}$ takes values in the category of chain complexes. In this case, $C^*(\boldsymbol{F}^\bullet)$ has the total differential formed by the differential of $\boldsymbol{F}^\bullet$ and (3.2).

Recall the definition of the compact dual $\boldsymbol{X}_{\mathcal{M}_{\mathcal{G}}}^{(c)}$ and of the embeddings $\boldsymbol{X}_{\mathcal{M}_{\mathcal{P}}}^{(c)} \to \boldsymbol{X}_{\mathcal{M}_{\mathcal{G}}}^{(c)}$ from the introduction. For a topological space X, a closed subset Y and a vector space V, let V_Y be the constant sheaf with stalk V on Y and let $(i_{Y\subseteq X})_* V_Y$ be its direct image on X. If $\mathbb{K}$ is either $\mathbb{R}$ or $\mathbb{C}$, let $\boldsymbol{A}(\mathcal{G}, \boldsymbol{K}_\infty^o, \mathbb{K})^{\mathcal{P}}$ be the functor which to $\mathcal{P}\in\mathfrak{P}$ associates the sheaf with $\mathcal{G}(\mathbb{A}_f)$-action

$$\left(i_{\boldsymbol{X}_{\mathcal{M}_{\mathcal{P}}}^{(c)}\subseteq \boldsymbol{X}_{\mathcal{M}_{\mathcal{G}}}^{(c)}}\right)_* C_c^\infty\left(\mathcal{P}(\mathbb{A}_f)\backslash\mathcal{G}(\mathbb{A}_f), \mathbb{K}\right)_{\boldsymbol{X}_{\mathcal{M}_{\mathcal{P}}}^{(c)}}.$$

For $\mathcal{P} \subseteq \mathcal{Q}$, $\boldsymbol{A}(\mathcal{G}, \boldsymbol{K}_\infty^o, \mathbb{K})^{\mathcal{P}\subseteq\mathcal{Q}}$ is defined by the inclusion

$$C_c^\infty(\mathcal{Q}(\mathbb{A}_f)\backslash\mathcal{G}(\mathbb{A}_f), \mathbb{K}) \subseteq C_c^\infty(\mathcal{P}(\mathbb{A}_f)\backslash\mathcal{G}(\mathbb{A}_f), \mathbb{K}),$$

followed by restriction from $\boldsymbol{X}^{(c)}_{\mathcal{M}_\mathcal{Q}}$ to $\boldsymbol{X}^{(c)}_{\mathcal{M}_\mathcal{P}}$. The group $\boldsymbol{K}_\infty^o \cap \mathcal{P}_o(\mathbb{R})$ acts on this complex by left translation, and the resulting action on hypercohomology factorizes over the quotient $\pi_0(\boldsymbol{K}_\infty^o \cap \mathcal{P}_o(\mathbb{R}))$. Recall the Borel map $I^*_{\mathcal{M}_\mathcal{G}(\mathbb{R}),\boldsymbol{K}_\infty^o} \to H^*(\mathcal{G}, \mathbb{C})^{\mathcal{G}(\mathbb{A}_f)}$ and the isomorphism $I^*_{\mathcal{M}_\mathcal{G},\boldsymbol{K}_\infty^o} \cong H^*(\boldsymbol{X}^{(c)}_{\mathcal{M}_\mathcal{G}})$ from the introduction.

With this notation, we can formulate our main result as follows:

Theorem 3.1. *There is a canonical isomorphism of $\mathcal{G}(\mathbb{A}_f)$- and $I^*_{\mathcal{M}_\mathcal{G}(\mathbb{R}),\boldsymbol{K}_\infty^o} \cong H^*(\boldsymbol{X}^{(c)}_{\mathcal{M}_\mathcal{G}})$-modules between $H_c^*(\mathcal{G}, \mathbb{C})_\mathcal{I}$ and the hypercohomology of the complex associated to the functor $\boldsymbol{A}(\mathcal{G}, \mathbb{C})^\mathcal{P}$*

$$H_c^*(\mathcal{G}, \mathbb{C})_\mathcal{I} \cong H^*(\boldsymbol{X}^{(c)}_{\mathcal{M}_\mathcal{G}}, C^*(\boldsymbol{A}(\mathcal{G}, \mathcal{C})))^{\pi_o(\boldsymbol{K}_\infty^o \cap \mathcal{P}_o(\mathbb{R}))}. \tag{3.4}$$

This isomorphism identifies the real subspace $H_c^p(\mathcal{G}, \mathbb{R})_\mathcal{I}$ with

$$i^p H^p(\boldsymbol{X}^{(c)}_{\mathcal{M}_\mathcal{G}}, C^*(\boldsymbol{A}(\mathcal{G}, \mathcal{C})))^{\pi_o(\boldsymbol{K}_\infty^o \cap \mathcal{P}_o(\mathbb{R}))}.$$

The proof of this theorem will occupy most of the remainder of this paper. We will now give some corollaries. Since the sheaf of $\mathcal{G}(\mathbb{A}_f)$-modules $\boldsymbol{A}(\mathcal{G}, \mathbb{C})^\mathcal{P}$ is annihilated by $\mathcal{I}_S$, we have the following result.

Corollary 3.2. *If S is a set of finite places of $\mathbb{Q}$, then $H^*(\mathcal{G}, \mathbb{C})_\mathcal{I}$ and $H_c^*(\mathcal{G}, \mathbb{C})_\mathcal{I}$ are annihilated by $\mathcal{I}_S$ (and not just a power of $\mathcal{I}_S$).*

The assertion about $H^*(\mathcal{G}, \mathbb{C})_\mathcal{I}$ follows from the result about cohomology with compact support by duality.

To evaluate the cohomology sheaves of the complex $C^*(\boldsymbol{A}(\mathcal{G}, \mathbb{C})^\bullet)$, we have to define some Steinberg-like $\mathcal{G}(\mathbb{A}_f)$-modules. Let

$$\mathfrak{V}^{\mathcal{G}(\mathbb{A}_f)}_{\mathcal{P}(\mathbb{A}_f)} = C^\infty(\mathcal{P}(\mathbb{A}_f)\backslash\mathcal{G}(\mathbb{A}_f), \mathbb{C}) \Big/ \sum_{\mathcal{Q}\supset\mathcal{P}} C^\infty(\mathcal{Q}(\mathbb{A}_f)\backslash\mathcal{G}(\mathbb{A}_f), \mathbb{C}), \tag{3.5}$$

and let $\check{\mathfrak{V}}^{\mathcal{G}(\mathbb{A}_f)}_{\mathcal{P}(\mathbb{A}_f)}$ be the dual of $\mathfrak{V}^{\mathcal{G}(\mathbb{A}_f)}_{\mathcal{P}(\mathbb{A}_f)}$. For instance, $\mathfrak{V}^{\mathcal{G}(\mathbb{A}_f)}_{\mathcal{G}(\mathbb{A}_f)}$ and $\check{\mathfrak{V}}^{\mathcal{G}(\mathbb{A}_f)}_{\mathcal{G}(\mathbb{A}_f)}$ are both isomorphic to the constant representation. Recall the definition of the subsets

$$\boldsymbol{U}_{\mathcal{M}_\mathcal{P}} = \boldsymbol{X}^{(c)}_{\mathcal{M}_\mathcal{P}} - \bigcup_{\mathcal{Q}\subset\mathcal{P}} \boldsymbol{X}^{(c)}_{\mathcal{M}_\mathcal{Q}}.$$

If V is a sheaf on $\boldsymbol{U}_{\mathcal{M}_\mathcal{P}}$, let $\left(i_{\boldsymbol{U}_{\mathcal{M}_\mathcal{P}} \subseteq \boldsymbol{X}^{(c)}_{\mathcal{M}_\mathcal{G}}}\right)_! V$ be its continuation by zero.

Corollary 3.3. *The i-th cohomology sheaf of the complex $C^*(\boldsymbol{A}(\mathcal{G}, \boldsymbol{K}_\infty^o, \mathbb{C}))$ is given by*

$$\bigoplus_{\substack{\mathcal{P}\in\mathfrak{P}\\ \dim\mathfrak{a}_{\mathcal{P}}^{\mathcal{G}}=i}} \left(i_{\boldsymbol{U}_{\mathcal{M}_{\mathcal{P}}}\subseteq\boldsymbol{X}^{(c)}_{\mathcal{M}_{\mathcal{G}}}}\right)_! \mathfrak{V}^{\mathcal{G}(\mathbb{A}_f)}_{\mathcal{P}(\mathbb{A}_f)}. \tag{3.6}$$

The hypercohomology spectral sequence degenerates, and the limit filtration $\mathrm{Fil}_i\, H_c^*(\mathcal{G},\mathbb{C})_{\mathcal{I}}$ *has quotients*

$$(\mathrm{Fil}_i/\mathrm{Fil}_{i-1})\, H_c^k(\mathcal{G},\mathbb{C})_{\mathcal{I}} \cong \bigoplus_{\substack{\mathcal{P}\in\mathfrak{P}\\ \dim\mathfrak{a}_{\mathcal{P}}^{\mathcal{G}}=i}} H_c^{k-i}(\boldsymbol{U}_{\mathcal{M}_{\mathcal{P}}})^{\pi_o(\boldsymbol{K}_\infty\cap\mathcal{P}_o(\mathbb{R}))}\otimes\mathfrak{V}^{\mathcal{G}(\mathbb{A}_f)}_{\mathcal{P}(\mathbb{A}_f)}, \tag{3.7}$$

where the isomorphism is an isomorphism of modules over $\mathcal{G}(\mathbb{A}_f)$. *This is the only ascending filtration of* $H_c^*(\mathcal{G},\mathbb{C})_{\mathcal{I}}$ *whose i-th quotient is of the form*

$$\bigoplus_{\substack{\mathcal{P}\in\mathfrak{P}\\ \dim\mathfrak{a}_{\mathcal{P}}^{\mathcal{G}}=i}} V_{\mathcal{P}}\otimes\mathfrak{V}^{\mathcal{G}(\mathbb{A}_f)}_{\mathcal{P}(\mathbb{A}_f)}.$$

Similarly, $H^*(\mathcal{G},\mathbb{C})_{\mathcal{I}}$ *has a descending filtration* Fil^i *with quotients*

$$\left(\mathrm{Fil}^i/\mathrm{Fil}^{i+1}\right) H_c^k(\mathcal{G},\mathbb{C})_{\mathcal{I}} \cong \bigoplus_{\substack{\mathcal{P}\in\mathfrak{P}\\ \dim\mathfrak{a}_{\mathcal{P}}^{\mathcal{G}}=i}} H^{k+\dim(\mathfrak{n}_{\mathcal{P}})}(\boldsymbol{U}_{\mathcal{M}_{\mathcal{P}}})^{\pi_o(\boldsymbol{K}_\infty\cap\mathcal{P}_o(\mathbb{R}))}\otimes\check{\mathfrak{V}}^{\mathcal{G}(\mathbb{A}_f)}_{\mathcal{P}(\mathbb{A}_f)}. \tag{3.8}$$

This is the only descending filtration of $H^*(\mathcal{G},\mathbb{C})_{\mathcal{I}}$ *whose i-th quotient is of the form*

$$\bigoplus_{\substack{\mathcal{P}\in\mathfrak{P}\\ \dim\mathfrak{a}_{\mathcal{P}}^{\mathcal{G}}=i}} V^{\mathcal{P}}\otimes\mathfrak{V}^{\mathcal{G}(\mathbb{A}_f)}_{\mathcal{P}(\mathbb{A}_f)}.$$

Proof. By Poincare duality, it suffices to prove the assertions about cohomology with compact support. The formula (3.6) is a consequence of the Solomon–Tits like Theorem 4.2 in the next section, which generalizes [4, §3]. The degeneration of the hypercohomology spectral sequence follows from Hodge theory and the fact that the restriction of an invariant (= harmonic) form on the compact dual of a Levi component of $\mathcal{G}$ to the compact dual of a smaller Levi component is again invariant.

The uniqueness assertion about the filtration of $H_c^*(\mathcal{G},\mathbb{C})_{\mathcal{I}}$ follows from the next proposition. □

Proposition 3.4. *Let S be a set which contains all nonarchimedean primes of $\mathbb{Q}$ with finitely many exceptions, and let $\mathcal{P}\neq\mathcal{Q}$ be parabolic subgroups of $\mathcal{G}$. Then the spaces of S-spherical vectors* ${\mathfrak{V}^{\mathcal{G}(\mathbb{A}_f)}_{\mathcal{P}(\mathbb{A}_f)}}^{\boldsymbol{K}_S}$ *and* ${\mathfrak{V}^{\mathcal{G}(\mathbb{A}_f)}_{\mathcal{Q}(\mathbb{A}_f)}}^{\boldsymbol{K}_S}$ *have finite length as representations of the group*

$$\prod_{\substack{v \text{ nonarchimedean}\\ v\notin S}} \mathcal{G}(\mathbb{Q}_v),$$

and their Jordan–Hölder series have mutually nonisomorphic quotients.

Proof. This is a consequence of [7, X.4.6.]. □

Unfortunately, Hilbert modular forms and SL_3 over imaginary fields provide examples where the filtration $\mathrm{Fil}_i\, H_c^*(\mathcal{G},\mathbb{C})_{\mathcal{I}}$ does not split in the category of $\mathcal{G}(\mathbb{A}_f)$-modules.

Since $\boldsymbol{K}_f$ was supposed to be good, we have $\mathcal{P}(\mathbb{A}_f)\boldsymbol{K}_f = \mathcal{G}(\mathbb{A}_f)$ for all parabolic subgroups $\mathcal{P}$. Therefore, $\mathfrak{V}_{\mathcal{P}(\mathbb{A}_f)}^{\mathcal{G}(\mathbb{A}_f)}$ has $\boldsymbol{K}_f$-spherical vectors only if $\mathcal{P} = \mathcal{G}$, and the only quotient of $\mathrm{Fil}_i\, H_c^*(\mathcal{G},\mathbb{C})_{\mathcal{I}}$ which has a $\boldsymbol{K}_f$-spherical vector is in dimension zero. We get the following corollary.

Corollary 3.5. *The natural maps*

$$H_c^*(\mathcal{G},\mathbb{C})^{\mathcal{G}(\mathbb{A}_f)} \to H_c^*(\mathcal{G},\mathbb{C})_{\mathcal{I}}^{\mathcal{G}(\mathbb{A}_f)} \to H_c^*(\mathcal{G},\mathbb{C})_{\mathcal{I}}^{\boldsymbol{K}_f}$$

are isomorphisms. (The first of these isomorphisms follows from the fact that the constant $\mathcal{G}(\mathbb{A}_f)$-representation is annihilated by $\mathcal{I}$.) Similarly, the maps

$$H^*(\mathcal{G},\mathbb{C})_{\mathcal{I}}^{\boldsymbol{K}_f} \to (H^*(\mathcal{G},\mathbb{C})_{\mathcal{I}})_{\mathcal{G}(\mathbb{A}_f)} \to H^*(\mathcal{G},\mathbb{C})_{\mathcal{G}(\mathbb{A}_f)}$$

are isomorphisms, where the subscript $_{\mathcal{G}(\mathbb{A}_f)}$ stands for the space of $\mathcal{G}(\mathbb{A}_f)$-coinvariants. Also, we have isomorphisms of $I^*_{\mathcal{M}_{\mathcal{G}}(\mathbb{R}),\boldsymbol{K}_\infty^\circ} \cong H^*(\boldsymbol{X}^{(c)}_{\mathcal{M}_{\mathcal{G}}})$-modules

$$H^*(\mathcal{G},\mathbb{C})_{\mathcal{G}(\mathbb{A}_f)} \cong H^*(\boldsymbol{U}_{\mathcal{M}_{\mathcal{G}}},\mathbb{C})$$

and

$$H_c^*(\mathcal{G},\mathbb{C})^{\mathcal{G}(\mathbb{A}_f)} \cong H_c^*(\boldsymbol{U}_{\mathcal{M}_{\mathcal{G}}},\mathbb{C}).$$

In particular, this establishes Theorem 1.1 of the introduction.

4 An adelic Borel–Serre–Solomon–Tits theorem

In this section we study the cohomology of the chain complexes associated to certain functors on $\mathfrak{P}$. Let us start with the easiest example. For parabolic subgroups $\mathcal{Q} \subseteq \mathcal{R}$, consider the contravariant functor

$$\boldsymbol{B}(\mathcal{Q},\mathcal{R})^{\mathcal{P}} = \begin{cases} \mathbb{C} & \text{if } \mathcal{Q} \subseteq \mathcal{P} \subseteq \mathcal{R} \\ \{0\} & \text{otherwise} \end{cases}$$

and the covariant functor

$$\boldsymbol{B}(\mathcal{Q},\mathcal{R})_{\mathcal{P}} = \begin{cases} \mathbb{C} & \text{if } \mathcal{Q} \subseteq \mathcal{P} \subseteq \mathcal{R} \\ \{0\} & \text{otherwise} \end{cases}$$

such that $\boldsymbol{B}(\mathcal{Q},\mathcal{R})^{\mathcal{P}\subseteq\tilde{\mathcal{P}}}$ and $\boldsymbol{B}(\mathcal{Q},\mathcal{R})_{\mathcal{P}\subseteq\tilde{\mathcal{P}}}$ are the identities if $\mathcal{Q} \subseteq \mathcal{P} \subseteq \tilde{\mathcal{P}} \subseteq \mathcal{R}$ and zero otherwise.

Lemma 4.1. *If $\mathcal{Q} \subset \mathcal{R}$, $C^*(\boldsymbol{B}(\mathcal{Q},\mathcal{R})^\bullet)$ and $C^*(\boldsymbol{B}(\mathcal{Q},\mathcal{R})_\bullet)$ are acyclic. If $\mathcal{Q} = \mathcal{R}$, then the only cohomology group of $C^*(\boldsymbol{B}(\mathcal{Q},\mathcal{R})^\bullet)$ is $\mathbb{C}$ in dimension $\dim \mathfrak{a}_{\mathcal{Q}}^{\mathcal{G}}$, and the only cohomology group of $C^*(\boldsymbol{B}(\mathcal{Q},\mathcal{R})_\bullet)$ is $\mathbb{C}$ in dimension $\dim \mathfrak{a}_o^{\mathcal{Q}}$.*

This is straightforward.

For a more interesting example, one takes the set of all $\mathbb{C}$-valued functions on $\mathcal{P}(\mathbb{Q})\backslash\mathcal{G}(\mathbb{Q})$ for $\boldsymbol{F}^{\mathcal{P}}$ together with the obvious inclusions as transition maps. The associated chain complex gives the reduced cohomology of the Tits building of $\mathcal{G}$ shifted by -1; hence by the Solomon–Tits theorem it has cohomology only in degree $\dim \mathfrak{a}_o^{\mathcal{G}}$. The related theorem in which continuous functions on $\mathcal{P}(\mathbb{Q}_v)\backslash\mathcal{G}(\mathbb{Q}_v)$ (with $\mathbb{Q}_v$-rational parabolic subgroups $\mathcal{P}$ which are standard with respect to a minimal $\mathbb{Q}_v$-rational parabolic subgroup) are considered has been proved by Borel and Serre [4, §3]. We need an adelic version of their result.

Theorem 4.2. *Let S be a set of places of $\mathbb{Q}$, and let $\mathcal{R}$ be a standard $\mathbb{Q}$-parabolic subgroup. Let $\boldsymbol{C}(\mathcal{G},\mathcal{R},\mathbb{A}_S)^\bullet$ be defined by*

$$\boldsymbol{C}(\mathcal{G},\mathcal{R},\mathbb{A}_S)^{\mathcal{P}} = \begin{cases} C^\infty(\mathcal{P}(\mathbb{A}_S)\backslash\mathcal{G}(\mathbb{A}_S)) & \text{if } \mathcal{P} \subseteq \mathcal{R} \\ \{0\} & \text{otherwise.} \end{cases}$$

(Recall our convention that C^∞-functions are supposed to be $\boldsymbol{K}_\infty$-finite.) Let the transition functions for $\boldsymbol{C}$ be given by the obvious inclusions. Then the complex $C^*(\boldsymbol{C}(\mathcal{G},\mathcal{R},\mathbb{A}_S)^\bullet)$ is acyclic in dimension $< \dim \mathfrak{a}_o^{\mathcal{G}}$.

Proof. The only difference to the situation considered by Borel and Serre is that we consider quotients of an adelic group by $\mathbb{Q}$-parabolic subgroups, whereas they consider quotients of the v-adic group by $\mathbb{Q}_v$-rational subgroups. Their method is flexible enough to cover our situation. To eliminate any possible doubt, let us give the modified proof.

Since $\boldsymbol{C}(\mathcal{G},\mathcal{P},\mathbb{A}_S)^\bullet$ is the inductive limit of its subfunctors $\boldsymbol{C}(\mathcal{G},\mathcal{P},\mathbb{A}_T)$ for finite T, it suffices to consider the case where S is a finite set of places of $\mathbb{Q}$. We will prove the following proposition.

Proposition 4.3. *Let S be a finite set of places of $\mathbb{Q}$, let B be a Banach space, and let $\mathcal{R} \in \mathfrak{P}$. Let $\tilde{\boldsymbol{C}}(\mathcal{R},\mathbb{A}_S)$ be given by spaces of B-valued continuous functions on flag varieties of $\mathcal{G}$*

$$\tilde{\boldsymbol{C}}(\mathcal{R},\mathbb{A}_S,B)^{\mathcal{P}} = \begin{cases} C(\mathcal{P}(\mathbb{A}_S)\backslash\mathcal{G}(\mathbb{A}_S),B) & \text{if } \mathcal{P} \subseteq \mathcal{R} \\ \{0\} & \text{otherwise} \end{cases}$$

with the obvious inclusions as transition homomorphisms. Then the complex $C^*(\tilde{\boldsymbol{C}}(\mathcal{R},\mathbb{A}_S,B)^\bullet)$ is acyclic in dimension $< \dim \mathfrak{a}_o^{\mathcal{G}}$.

For finite S, the theorem follows from the proposition, since $\boldsymbol{C}(\mathcal{G},\mathcal{R},\mathbb{A}_S)$ is the inductive limit of its subfunctors $\boldsymbol{C}(\mathcal{G},\mathcal{R},\mathbb{A}_S)e$ over idempotents e of the convolution algebra $C^\infty(\boldsymbol{K}_S)$. But $\boldsymbol{C}(\mathcal{G},\mathcal{R},\mathbb{A}_S)e = \tilde{\boldsymbol{C}}(\mathcal{R},\mathbb{A}_S,\mathbb{C})e$ is a direct summand of $\tilde{\boldsymbol{C}}(\mathcal{R},\mathbb{A}_S,\mathbb{C})$. □

Proof. We proceed by induction on the cardinality of S, starting with the case $S = \emptyset$. For this case, we have $\tilde{\boldsymbol{C}}(\mathcal{R}, \mathbb{A}_\emptyset, B)^\bullet : \ = \boldsymbol{B}(\mathcal{P}_o, \mathcal{R})^\bullet \otimes B$ and apply Lemma 4.1.

Let $v \in S$ be such that the proposition has been verified for $S \setminus \{v\}$ and arbitrary $\mathcal{R}$ and B. Let $\mathcal{P}_v \subseteq \mathcal{P}_o$ be a minimal $\mathbb{Q}_v$-parabolic subgroup, and let $\mathcal{A}_v \subset \mathcal{P}_v$ be a maximal $\mathbb{Q}_v$-split torus. Let $w_0, \ldots, w_N$ be an enumeration of the elements of the Weyl group $W(\mathcal{A}_v, \mathcal{G}(\mathbb{Q}_v))$ such that $\ell(w_i) \leq \ell(w_j)$ if $i < j$, where $\ell(w)$ is the length of w. Let

$$C(w) = \mathcal{P}_v(\mathbb{Q}_v)\backslash\mathcal{P}_v(\mathbb{Q}_v)w\mathcal{P}_v(\mathbb{Q}_v) \subset \mathcal{P}_v(\mathbb{Q}_v)\backslash\mathcal{G}(\mathbb{Q}_v)$$

be the Schubert cell associated to w, and let $E_i = \bigcup_{j=0}^{i} C(w_j)$. Let Δ_v and $\Delta_v^{\mathcal{P}}$ be defined like Δ_o and $\Delta_o^{\mathcal{P}}$, but with $\mathcal{P}_o$ replaced by $\mathcal{P}_v$. For $\alpha \in \Delta_v$, let s_α be the reflection belonging to α. Let

$$\pi_{\mathcal{P}} \colon \mathcal{P}_v(\mathbb{Q}_v)\backslash\mathcal{G}(\mathbb{Q}_v) \to \mathcal{P}(\mathbb{Q}_v)\backslash\mathcal{G}(\mathbb{Q}_v)$$

be the projection. We have the following consequence of the Bruhat decomposition as given in [4, 2.4.].

Lemma 4.4. *Let $0 \leq i \leq N$. If $\mathcal{P} \supseteq \mathcal{P}_v$ is a $\mathbb{Q}_v$-parabolic subgroup such that $\ell(s_\alpha w_i) > \ell(w_i)$ for all $\alpha \in \Delta_o^{\mathcal{P}}$, then $\pi_{\mathcal{P}}$ induces an isomorphism*

$$C(w_i) \cong \pi_{\mathcal{P}}(C(w_i)) = \pi_{\mathcal{P}}(E_i) - \pi_{\mathcal{P}}(E_{i-1}).$$

Otherwise, we have $\pi_{\mathcal{P}}(E_i) = \pi_{\mathcal{P}}(E_{i-1})$.

Let $\mathrm{Fil}^i\, \tilde{\boldsymbol{C}}(\mathcal{R}, \mathbb{A}_S, B)^{\mathcal{P}}$ be the set of all $f \in \tilde{\boldsymbol{C}}(\mathcal{R}, \mathbb{A}_S, B)^{\mathcal{P}}$ which vanish on

$$\big(\mathcal{P}(\mathbb{A}_{S\setminus\{v\}})\backslash\mathcal{G}(\mathbb{A}_{S\setminus\{v\}})\big) \times \pi_{\mathcal{P}}(E_i). \tag{4.1}$$

This is a subfunctor of $\tilde{\boldsymbol{C}}(\mathcal{R}, \mathbb{A}_S, B)^\bullet$. Let us consider $0 \leq i \leq N$. If there exists no $\mathbb{Q}$-parabolic subgroup $\mathcal{Q} \supseteq \mathcal{P}_v$ such that $\ell(s_\alpha w_i) > \ell(w_i)$ for all $\alpha \in \Delta_v^{\mathcal{Q}}$, then Lemma 4.4 implies $\mathrm{Fil}^{i-1}\, \tilde{\boldsymbol{C}}(\mathcal{R}, \mathbb{A}_S, B)^\bullet = \mathrm{Fil}^i\, \tilde{\boldsymbol{C}}(\mathcal{R}, \mathbb{A}_S, B)^\bullet$. Otherwise, let $\mathcal{Q}_i$ be the largest $\mathbb{Q}$-parabolic subgroup with this property. Let $C(C(w), B)$ be the Banach space of continuous B-valued functions on $C(w)$, and let $\overline{C_c(C(w), B)} \subset C(C(w), B)$ be the closure of the set of compactly supported functions. Identifying B-valued continuous functions on (4.1) with $C(\pi_{\mathcal{P}}(E_i), B)$-valued continuous functions on

$$\mathcal{P}(\mathbb{A}_{S\setminus\{v\}})\backslash\mathcal{G}(\mathbb{A}_{S\setminus\{v\}})$$

and using the isomorphism

$$\ker\big(C(\pi_{\mathcal{P}}(E_i), B) \to C(\pi_{\mathcal{P}}(E_{i-1}), B)\big) = \begin{cases} \{0\} & \text{if } \mathcal{P} \not\subset \mathcal{Q}_i \\ \overline{C_c(C(w_i), B)} & \text{if } \mathcal{P} \subset \mathcal{Q}_i, \end{cases}$$

we get an isomorphism

$$\big(\mathrm{Fil}^{i-1} / \mathrm{Fil}^i\big)\, \tilde{\boldsymbol{C}}(\mathcal{R}, \mathbb{A}_S, B)^{\mathcal{P}} \cong \tilde{\boldsymbol{C}}(\mathcal{R} \cap \mathcal{Q}_i, \mathbb{A}_{S-\{v\}}, \overline{C_c(C(w_i), B)}),$$

and the induction argument is complete. □

Generalising the definition of $\mathfrak{V}_{\mathcal{P}(\mathbb{A}_f)}^{\mathcal{G}(\mathbb{A}_f)}$ in the third section, we define

$$\mathfrak{V}_{\mathcal{P}(\mathbb{A}_S)}^{\mathcal{G}(\mathbb{A}_S)} = C^\infty\left(\mathcal{P}(\mathbb{A}_S)\backslash\mathcal{G}(\mathbb{A}_S)\right) \Big/ \sum_{\mathcal{Q}\supset\mathcal{P}} C^\infty\left(\mathcal{Q}(\mathbb{A}_S)\backslash\mathcal{G}(\mathbb{A}_S)\right) \tag{4.2}$$

where it is understood that if S contains the archimedean place, then induction at this place is $(\mathfrak{g}, \boldsymbol{K})$-module induction. Let $\check{\mathfrak{V}}_{\mathcal{P}(\mathbb{A}_S)}^{\mathcal{G}(\mathbb{A}_S)}$ be the $\boldsymbol{K}_S$-finite dual of $\mathfrak{V}_{\mathcal{P}(\mathbb{A}_S)}^{\mathcal{G}(\mathbb{A}_S)}$. We put $\mathfrak{S}t_{\mathcal{G}(\mathbb{A}_S)} = \mathfrak{V}_{\mathcal{G}(\mathbb{A}_S)}^{\mathcal{P}_o(\mathbb{A}_S)}$ and $\check{\mathfrak{S}}t_{\mathcal{G}(\mathbb{A}_S)} = \check{\mathfrak{V}}_{\mathcal{G}(\mathbb{A}_S)}^{\mathcal{P}_o(\mathbb{A}_S)}$. These can be considered as Steinberg-like modules, although they are highly non-irreducible unless S consists of a single place v at which $\mathcal{P}_o$ is also a minimal $\mathbb{Q}_v$-parabolic subgroup.

If we choose Haar measures on $\mathcal{G}(\mathbb{A})$ and $\mathcal{P}_o(\mathbb{A})$, then the dual of

$$C^\infty(\mathbb{P}(\mathbb{A})\backslash\mathcal{G}(\mathbb{A})) = \operatorname{Ind}_{\mathcal{P}}^{\mathcal{G}} \mathbb{C}$$

can be identified with $\operatorname{Ind}_{\mathcal{P}}^{\mathcal{G}} \mathbb{C}_{2\rho_o}$. This allows us to view

$$\operatorname{Ind}_{\mathcal{P}}^{\mathcal{G}} \check{\mathfrak{S}}t_{\mathcal{L}_{\mathcal{P}}(\mathbb{A})} \otimes \mathbb{C}_{2\rho_{\mathcal{P}}}$$

as a submodule of $\operatorname{Ind}_{\mathcal{P}_o}^{\mathcal{G}} \mathbb{C}_{2\rho_o}$. It is the orthogonal complement of

$$\sum_{\mathcal{Q}\subset\mathcal{P}} C^\infty(\mathcal{Q}(\mathbb{A})\backslash\mathcal{G}(\mathbb{A})),$$

hence it decreases if $\mathcal{P}$ increases. This allows us to define

$$\boldsymbol{D}(\mathcal{G})^{\mathcal{P}} = \left\{ \begin{matrix} \operatorname{Ind}_{\mathcal{P}}^{\mathcal{G}} \check{\mathfrak{S}}t_{\mathcal{L}_{\mathcal{P}}(\mathbb{A})} \otimes \mathbb{C}_{2\rho_{\mathcal{P}}} & \text{if } \mathcal{P} \neq \mathcal{P}_o \\ \sum_{\mathcal{P}\in\mathfrak{P}} \operatorname{Ind}_{\mathcal{P}}^{\mathcal{G}} \check{\mathfrak{S}}t_{\mathcal{L}_{\mathcal{P}}(\mathbb{A})} \otimes \mathbb{C}_{2\rho_{\mathcal{P}}} & \text{if } \mathcal{P} = \mathcal{P}_o \end{matrix} \right\} \subset \operatorname{Ind}_{\mathcal{P}_o}^{\mathcal{G}} \mathbb{C}_{2\rho_o}. \tag{4.3}$$

Theorem 4.5. *If* $\dim \mathfrak{a}_o^{\mathcal{G}} > 0$, *then* $C^*(\boldsymbol{D}(\mathcal{G})^\bullet)$ *is acyclic.*

Proof. $\boldsymbol{D}(\mathcal{G})^\bullet \subset \boldsymbol{B}(\mathcal{P}_o, \mathcal{G})^\bullet \otimes \operatorname{Ind}_{\mathcal{P}_o}^{\mathcal{G}} \mathbb{C}_{2\rho_o}$ is the orthogonal complement of

$$\boldsymbol{M}_\bullet \subset \boldsymbol{B}(\mathcal{P}_o, \mathcal{G})_\bullet \otimes \operatorname{Ind}_{\mathcal{P}_o}^{\mathcal{G}} \mathbb{C},$$

where

$$\boldsymbol{M}_{\mathcal{P}} = \left\{ \begin{matrix} \sum_{\substack{\mathcal{R}\subseteq\mathcal{P} \\ \dim \mathfrak{a}_o^{\mathcal{R}}=1}} C^\infty(\mathcal{R}(\mathbb{A})\backslash\mathcal{G}(\mathbb{A})) & \text{if } \mathcal{P} \neq \mathcal{P}_o \\ \mathbb{C} = \bigcap_{\substack{\mathcal{R}\in\mathfrak{P} \\ \dim \mathfrak{a}_o^{\mathcal{R}}=1}} C^\infty(\mathcal{R}(\mathbb{A})\backslash\mathcal{G}(\mathbb{A})) & \text{otherwise} \end{matrix} \right\} \subset C^\infty(\mathcal{P}_o(\mathbb{A})\backslash\mathcal{G}(\mathbb{A})).$$

By Lemma 4.1, it suffices to show that $C^*(\boldsymbol{M}_\bullet)$ is acyclic. Let

$$\tilde{\boldsymbol{M}}_{\mathcal{P}} = \begin{cases} \boldsymbol{M}_{\mathcal{P}} & \text{if } \mathcal{P} \neq \mathcal{P}_o \\ \{0\} & \text{if } \mathcal{P} = \mathcal{P}_o. \end{cases}$$

Since $\mathbb{C} \subseteq \boldsymbol{M}_{\mathcal{P}_o} \subseteq H^1(C^*(\tilde{\boldsymbol{M}}_\bullet))$, the acyclicity of $C^*(\boldsymbol{M}_\bullet)$ and the theorem will follow if we show that $C^*(\tilde{\boldsymbol{M}}_\bullet)$ has only one one-dimensional cohomology space in dimension one.

We will reduce this to Theorem 4.2 by introducing a functor of two variables $\boldsymbol{N}_\bullet^\bullet$ and using the spectral sequence for its double complex. We define $\boldsymbol{N}_\bullet^\bullet$ by

$$\boldsymbol{N}_{\mathcal{P}}^{\mathcal{Q}} = \begin{cases} \{0\} & \text{if } \mathcal{Q} \not\subseteq \mathcal{P} \text{ or if } \mathcal{Q} = \mathcal{P}_o \\ C^\infty(\mathcal{Q}(\mathbb{A})\backslash\mathcal{G}(\mathbb{A})) & \text{otherwise.} \end{cases}$$

It is a consequence of Theorem 4.2 (applied to $\boldsymbol{C}(\mathcal{G}, \mathcal{L}_{\mathcal{G}}, \mathbb{A})$) that

$$H^k(C^*(\boldsymbol{N}_{\mathcal{P}}^\bullet)) = \begin{cases} \{0\} & \text{if } k \neq \dim \mathfrak{a}_o^{\mathcal{G}} - 1 \\ \tilde{\boldsymbol{M}}_{\mathcal{P}} & \text{if } k = \dim \mathfrak{a}_o^{\mathcal{G}} - 1. \end{cases}$$

Since

$$\boldsymbol{N}_\bullet^{\mathcal{Q}} = \begin{cases} \{0\} & \text{if } \mathcal{Q} = \mathcal{P}_o \\ \boldsymbol{B}(\mathcal{Q}, \mathcal{G})_\bullet \otimes C^\infty(\mathcal{Q}(\mathbb{A})\backslash\mathcal{G}(\mathbb{A})) & \text{if } \mathcal{Q} \supset \mathcal{P}_o, \end{cases}$$

Lemma 4.1 implies

$$H^l\left(C^*(\boldsymbol{N}_\bullet^{\mathcal{Q}})\right) = \begin{cases} \{0\} & \text{if } l \neq \dim \mathfrak{a}_o^{\mathcal{G}} \text{ or } \mathcal{Q} \neq \mathcal{G} \\ \mathbb{C} & \text{if } l = \dim \mathfrak{a}_o^{\mathcal{G}} \text{ and } \mathcal{Q} = \mathcal{G}. \end{cases}$$

Combining these two facts, we get

$$H^k\left(C^*(\tilde{\boldsymbol{M}}_\bullet)\right) = H^{k+\dim \mathfrak{a}_o^{\mathcal{G}}-1}\left(C^*(\boldsymbol{N}_\bullet^\bullet)\right) = \begin{cases} \{0\} & \text{if } k \neq 1 \\ \mathbb{C} & \text{if } k = 1. \end{cases}$$

As was mentioned earlier, this implies the theorem. □

We complete this section with a rather elementary lemma. For a parabolic subgroup $\mathcal{R}$ of $\mathcal{G}$, let

$$\boldsymbol{E}(\mathcal{R})^{\mathcal{P}*} = \begin{cases} \Lambda^*(\check{\mathfrak{a}}_{\mathcal{R}}^{\mathcal{P}}) & \text{if } \mathcal{P} \supseteq \mathcal{R} \\ \{0\} & \text{if } \mathcal{P} \not\supseteq \mathcal{R}. \end{cases} \tag{4.4}$$

The transition homomorphism $\boldsymbol{E}(\mathcal{R})^{\tilde{\mathcal{P}}\subseteq\mathcal{P}*}$ is given by the projection $\check{\mathfrak{a}}_{\mathcal{R}}^{\mathcal{P}} \to \check{\mathfrak{a}}_{\mathcal{R}}^{\tilde{\mathcal{P}}}$. $\boldsymbol{E}(\mathcal{R})^{\bullet*}$ is a functor from $\mathfrak{P}$ into the category of graded vector spaces.

Lemma 4.6. *The projection*

$$\boldsymbol{E}(\mathcal{R})^{\mathcal{G}*} = \Lambda^*(\check{\mathfrak{a}}_{\mathcal{R}}^{\mathcal{G}}) \to \det \check{\mathfrak{a}}_{\mathcal{R}}^{\mathcal{G}}[-\dim \check{\mathfrak{a}}_{\mathcal{R}}^{\mathcal{G}}]$$

defines an isomorphism on cohomology

$$H^*\left(C^*\left(\boldsymbol{E}(\mathcal{R})^{\bullet*}\right)\right) \cong \det \check{\mathfrak{a}}_{\mathcal{R}}^{\mathcal{G}}[-\dim \check{\mathfrak{a}}_{\mathcal{R}}^{\mathcal{G}}].$$

By the determinant of a finite dimensional vector space, we understand its highest exterior power.

Proof. Let $\mathcal{R}_1, \ldots, \mathcal{R}_k$ be the parabolic subgroups containing $\mathcal{R}$ with the property that $\dim \mathfrak{a}_{\mathcal{R}}^{\mathcal{R}_i} = 1$. Then

$$C^*\left(\boldsymbol{E}(\mathcal{R})^{\bullet *}\right) \cong \bigotimes_{i=1}^{k} \left((\mathbb{C} \oplus \check{\mathfrak{a}}_{\mathcal{R}}^{\mathcal{R}_i}) \to \mathbb{C}\right),$$

proving the lemma. □

5 The space of automorphic forms

It is known that $H^*(\mathcal{G}, \mathbb{C})$ can be evaluated by using the cohomology of the de Rham complex, which is isomorphic to the standard complex for evaluating the $(\mathfrak{m}_{\mathcal{G}}, \boldsymbol{K}_\infty^o)$-cohomology $H^*_{(\mathfrak{m}_{\mathcal{G}}, \boldsymbol{K}_\infty)}(C^\infty(\mathcal{A}_{\mathcal{G}}(\mathbb{R})^+\mathcal{G}(\mathbb{Q})\backslash\mathcal{G}(\mathbb{A})))$. Let

$$C^\infty_{\mathrm{umg}}(\mathcal{A}_{\mathcal{G}}(\mathbb{R})^+\mathcal{G}(\mathbb{Q})\backslash\mathcal{G}(\mathbb{A})) \subset C^\infty(\mathcal{A}_{\mathcal{G}}(\mathbb{R})^+\mathcal{G}(\mathbb{Q})\backslash\mathcal{G}(\mathbb{A})) \tag{5.1}$$

be the subspace of functions of uniformly moderate growth. Let $\mathcal{J} = \mathfrak{U}(\mathfrak{g})\mathfrak{g} \cap \mathfrak{Z}(\mathfrak{g})$ be the annihilator of the constant representation in $\mathfrak{Z}(\mathfrak{g})$. Let

$$A_{\mathcal{J}} \colon = \left\{ f \in C^\infty_{\mathrm{umg}}(\mathcal{A}_{\mathcal{G}}(\mathbb{R})^+\mathcal{G}(\mathbb{Q})\backslash\mathcal{G}(\mathbb{A})) \middle| \, \mathcal{J}^n f = \{0\} \text{ for } n \gg 0 \right\}. \tag{5.2}$$

Borel has verified that the inclusion (5.1) defines an isomorphism on cohomology and conjectured that the inclusion $A_{\mathcal{J}} \subset C^\infty_{\mathrm{umg}}$ also defines an isomorphism on cohomology with constant coefficients. After partial results by Casselman, Harder, and Speh, this has been verified in [9], where we denoted C^∞_{umg} by S_∞ and $A_{\mathcal{J}}$ by $\mathfrak{Fin}_{\mathcal{J}} S_\infty$ since we worked in a more general situation.

Let S be a set of finite primes which contains all but finitely many primes. It is a consequence of well-known finiteness properties of the space of automorphic forms (cf. [11, Proposition 2.3]) that the space of $\boldsymbol{K}_S$-spherical vectors $A_{\mathcal{J}}^{\boldsymbol{K}_S} \subset A_{\mathcal{J}}$ has a decomposition into associated Hecke eigenspaces

$$A_{\mathcal{J}}^{\boldsymbol{K}_S} = \coprod_{\tilde{\mathcal{I}}} A_{\mathcal{J},\tilde{\mathcal{I}}}^{\boldsymbol{K}_S},$$

where the sum is over maximal ideals $\tilde{\mathcal{I}} \subset \mathfrak{H}_S$ and

$$A_{\mathcal{J},\tilde{\mathcal{I}}}^{\boldsymbol{K}_S} = \left\{ f \in A_{\mathcal{J}} \middle| \tilde{\mathcal{I}}^n f = \{0\} \text{ for } n \gg 0 \right\}.$$

It is clear from the proven Borel conjecture that the cohomology of $A_{\mathcal{J},\tilde{\mathcal{I}}}^{\boldsymbol{K}_S}$ is isomorphic to the space of $\boldsymbol{K}_S$-spherical vectors in $H^*(\mathcal{G}, \mathbb{C})$ which are annihilated by a power of $\tilde{\mathcal{I}}$. Recall the maximal ideal $\mathcal{I}_S \subset \mathfrak{H}_S$, which is the annihilator of the constant representation. We put

$$A_{\mathcal{J},\mathcal{I}} \colon = \operatorname*{colim}_S A_{\mathcal{J},\mathcal{I}_S}^{\boldsymbol{K}_S}. \tag{5.3}$$

The aim of this section is to study $A_{\mathcal{J},\mathcal{I}}$.

There are two methods available for studying the space of automorphic forms. One method is to define a filtration on the space of automorphic forms, and to show that its quotients are spanned by principal values of cuspidal and residual Eisenstein series. This method was used in [9]. It is particularly useful in a general situation, where one has only the facts proved in Langlands' book [15] available. The second method, which was proposed by Harder in [14] before [9] was written, is to generate the space of automorphic forms by the coefficients of the Laurent expansions of cuspidal Eisenstein series at a certain point. In [11], we derived from the result of [9] that this procedure really gives the space of all automorphic forms. This method gives a complete description of the space of automorphic forms (and not just the quotients of a filtration), but is useful only if the precise structure of the singularities of the cuspidal Eisenstein series near the point where they have to be evaluated is known. For the Eisenstein series which contribute to $A_{\mathcal{J},\mathcal{I}}$, we are in the fortunate situation to have such information available. We will therefore generate the space $A_{\mathcal{J},\mathcal{I}}$ by cuspidal Eisenstein series. At the beginning, the procedure will be quite similar to the methods used by Speh in [22]. However, Speh studied only a certain subspace of $A_{\mathcal{J},\mathcal{I}}$, which was sufficient for her examples of the noninjectivity of the Borel map, and for which only Eisenstein series depending on one parameter were needed.

Let $\mathcal{P}$ be a standard parabolic subgroup. Recall the standard height function $H_{\mathcal{P}}\colon \mathcal{G}(\mathbb{A}) \to \mathfrak{a}_{\mathcal{P}}$, which is defined by

$$\langle H_{\mathcal{P}}(g), \chi\rangle = \sum_{v} \log |\chi(p_v)|_v \,, \tag{5.4}$$

where $g = pk$ with $p \in \mathcal{P}(\mathbb{A})$ and $k \in \boldsymbol{K}$. The scalar product $\langle .,.\rangle$ on the left side is the pairing between $\mathfrak{a}_{\mathcal{P}}$ and $\check{\mathfrak{a}}_{\mathcal{P}}$, and $\chi \in X^*(\mathcal{P}) \subset \check{\mathfrak{a}}_{\mathcal{P}}$. It is clear that (5.4) characterizes $H_{\mathcal{P}}(g)$ uniquely, and that $H_{\mathcal{P}}(g)$ does not depend on the choice of the Iwasawa decomposition $g = pk$. If $\mathcal{Q} \supseteq \mathcal{P}$, then $H_{\mathcal{Q}}(g)$ is the projection of $H_{\mathcal{P}}(g)$ to $\mathfrak{a}_{\mathcal{Q}}$.

We have to recall a few facts about the Eisenstein series starting from the constant representation of a Levi component. Proofs can be found in [10, Lemma 2.7], although the results about the Eisenstein series were almost certainly known previously. If $\phi \in C^\infty(\mathcal{P}(\mathbb{A})\backslash\mathcal{G}(\mathbb{A}))$, the Eisenstein series starting from ϕ is defined by

$$E^{\mathcal{G}}_{\mathcal{P}}(\phi, \lambda) = \sum_{\gamma\in\mathcal{P}(\mathbb{Q})\backslash\mathcal{G}(\mathbb{Q})} \phi(\gamma g)\, \mathrm{e}^{\langle\lambda+\rho_{\mathcal{P}}, H_{\mathcal{P}}(\gamma g)\rangle} \,. \tag{5.5}$$

This series converges for sufficiently regular $\Re\lambda$ in the positive Weyl chamber, and has an analytic continuation to $\lambda \in (\check{\mathfrak{a}}_{\mathcal{P}})_{\mathbb{C}}$. The singular hyperplanes of this function which cross through $\rho_{\mathcal{P}}$ are precisely the hyperplanes $\langle\lambda - \rho_{\mathcal{P}}, \check{\alpha}\rangle = 0$, where $\alpha \in \Delta_{\mathcal{P}}$ and $\check{\alpha}$ is the corresponding coroot. The residues may be described as follows. Let for $\lambda \in (\check{\mathfrak{a}}_{\mathcal{P}})_{\mathbb{C}}$

$$q_{\mathcal{P}}^{\mathcal{Q}}(\lambda) = \prod_{\alpha \in \Delta_{\mathcal{P}}^{\mathcal{Q}}} \langle \check{\alpha}, \lambda - \rho_{\mathcal{P}} \rangle.$$

Then the function $q_{\mathcal{P}}^{\mathcal{Q}}(\lambda) E_{\mathcal{P}}^{\mathcal{G}}(\phi, \lambda)$ is regular on an open dense subset of $\rho_{\mathcal{P}}^{\mathcal{Q}} + (\check{\mathfrak{a}}_{\mathcal{Q}})_{\mathbb{C}}$. Its restriction to $\rho_{\mathcal{P}}^{\mathcal{Q}} + (\check{\mathfrak{a}}_{\mathcal{Q}})_{\mathbb{C}}$ can be described as follows. If $\phi \in C^{\infty}(\mathcal{P}(\mathbb{A}) \backslash \mathcal{G}(\mathbb{A}))$, then

$$\mathrm{e}^{\langle H_{\mathcal{P}}(\cdot), 2\rho_{\mathcal{P}} \rangle} \, \phi(\cdot) \in \mathrm{Ind}_{\mathcal{P}}^{\mathcal{G}} \, \mathbb{C}_{2\rho_{\mathcal{P}}},$$

and let $\mathbb{C}_{2\rho_{\mathcal{P}}}$ be the one-dimensional vector space on which $p \in \mathcal{P}(\mathbb{A})$ acts by multiplication by $\mathrm{e}^{\langle H_{\mathcal{P}}(p), 2\rho_{\mathcal{P}} \rangle}$. There exists a unique nonvanishing homomorphism

$$\tau_{\mathcal{P}}^{\mathcal{Q}} \colon \mathrm{Ind}_{\mathcal{P}}^{\mathcal{G}} \, \mathbb{C}_{2\rho_{\mathcal{P}}} \to \mathrm{Ind}_{\mathcal{Q}}^{\mathcal{G}} \, \mathbb{C}_{2\rho_{\mathcal{Q}}}$$

with the following property. For generic $\vartheta \in \check{\mathfrak{a}}_{\mathcal{Q}}$ we have

$$\left(q_{\mathcal{P}}^{\mathcal{Q}}(\cdot) E_{\mathcal{P}}^{\mathcal{G}}(\cdot) \right) (\phi, \vartheta + \rho_{\mathcal{P}}^{\mathcal{Q}}) = E_{\mathcal{Q}}^{\mathcal{G}} \left(\mathrm{e}^{-\langle 2\rho_{\mathcal{Q}}, H_{\mathcal{Q}}(\cdot) \rangle} \, \tau_{\mathcal{P}}^{\mathcal{Q}} \left(\mathrm{e}^{\langle 2\rho_{\mathcal{P}}, H_{\mathcal{P}}(\cdot) \rangle} \, \phi \right), \vartheta \right). \quad (5.6)$$

It is easy to verify

$$\tau_{\mathcal{Q}}^{\mathcal{R}} \tau_{\mathcal{P}}^{\mathcal{Q}} = \tau_{\mathcal{P}}^{\mathcal{R}}$$

and to see that $\tau_{\mathcal{P}}^{\mathcal{Q}}$ is independent of $\boldsymbol{K}_f$.

Let $S(\check{\mathfrak{a}}_o^{\mathcal{G}})$ be the symmetric algebra of $\check{\mathfrak{a}}_o^{\mathcal{G}}$. It can be identified with the algebra of differential operators with constant coefficients on $\check{\mathfrak{a}}_o^{\mathcal{G}}$. After we choose a basis for $\check{\mathfrak{a}}_o^{\mathcal{G}}$, we have elements $\frac{\partial^{\alpha}}{\partial \lambda^{\alpha}} \in S(\check{\mathfrak{a}}_o^{\mathcal{G}})$ for any multi-index $\alpha = (\alpha_1, \ldots, \alpha_{\dim \check{\mathfrak{a}}_o^{\mathcal{G}}})$. Elements of $S(\check{\mathfrak{a}}_o^{\mathcal{G}})$ can also be viewed as polynomials on $\mathfrak{a}_{\mathcal{P}}^{\mathcal{G}} \subset \mathfrak{a}_{\mathcal{P}}$. Let H^{α} be the polynomial in $H \in \mathfrak{a}_{\mathcal{P}}$ belonging to $\frac{\partial^{\alpha}}{\partial \lambda^{\alpha}}$. We define a $\mathcal{G}(\mathbb{A}_f)$-action on

$$S(\check{\mathfrak{a}}_{\mathcal{P}}^{\mathcal{G}}) \otimes C^{\infty}(\mathcal{P}(\mathbb{A}) \backslash \mathcal{G}(\mathbb{A})) \quad (5.7)$$

by

$$\left(h \left(\frac{\partial^{\alpha}}{\partial \lambda^{\alpha}} \otimes \phi \right) \right)(g) =$$
$$\sum_{\alpha = \beta + \gamma} \left(\prod_{i=1}^{\dim \mathfrak{a}_{\mathcal{P}}^{\mathcal{G}}} \frac{\alpha_i!}{\beta_i! \gamma_i!} \right) \frac{\partial^{\beta}}{\partial \lambda^{\beta}} \otimes \left(\left(H_{\mathcal{P}}(gh) - H_{\mathcal{P}}(g) \right)^{\gamma} \mathrm{e}^{2\langle \rho_{\mathcal{P}}, H_{\mathcal{P}}(gh) - H_{\mathcal{P}}(g) \rangle} \, \phi(gh) \right) \quad (5.8)$$

for $h \in \mathcal{G}(\mathbb{A}_f)$. In a similar way, one obtains a $(\mathfrak{g}, \boldsymbol{K}_{\infty})$-module structure on (5.7) by taking the differential of the $\mathcal{G}(\mathbb{R})$-action which would be given by (5.8) if there was no condition of K_{∞}-finiteness for elements of $C^{\infty}(\mathcal{P}(\mathbb{A}) \backslash \mathcal{G}(\mathbb{A}))$. Let a $\mathcal{P}(\mathbb{A})$-action on $S(\check{\mathfrak{a}}_{\mathcal{P}}^{\mathcal{G}})$ be defined by

$$p \colon D \to \mathrm{e}^{-\langle H_{\mathcal{P}}(p), \cdot \rangle} \, D \, \mathrm{e}^{\langle H_{\mathcal{P}}(p), \cdot \rangle}.$$

At the infinite place, the $\mathcal{P}(\mathbb{R})$-action gives rise to the structure of a $(\mathfrak{p}, \boldsymbol{K}_\infty \cap \mathcal{P}(\mathbb{R}))$-module. There is a homomorphism of $(\mathfrak{p}, \boldsymbol{K}_\infty \cap \mathcal{P}(\mathbb{R}), \mathcal{P}(\mathbb{A}_f))$-modules

$$\begin{aligned} S(\check{\mathfrak{a}}_{\mathcal{P}}^{\mathcal{G}}) \otimes C^\infty(\mathcal{P}(\mathbb{A})\backslash\mathcal{G}(\mathbb{A})) &\to S(\check{\mathfrak{a}}_{\mathcal{P}}^{\mathcal{G}}) \otimes \mathbb{C}_{2\rho_{\mathcal{P}}} \\ D \otimes \phi(g) &\to D \otimes \phi(1) \end{aligned}$$

which defines an isomorphism

$$S(\check{\mathfrak{a}}_{\mathcal{P}}^{\mathcal{G}}) \otimes C^\infty(\mathcal{P}(\mathbb{A})\backslash\mathcal{G}(\mathbb{A})) \to \operatorname{Ind}_{\mathcal{P}}^{\mathcal{G}} S(\check{\mathfrak{a}}_{\mathcal{P}}^{\mathcal{G}}) \otimes \mathbb{C}_{2\rho_{\mathcal{P}}}. \tag{5.9}$$

Using this isomorphism and the regularity of $q_{\mathcal{P}}^{\mathcal{G}}(\cdot)E_{\mathcal{P}}^{\mathcal{G}}(\phi, \cdot)$ at $\rho_{\mathcal{P}}$, we get a homomorphism of $(\mathfrak{g}, \boldsymbol{K}_\infty, \mathcal{G}(\mathbb{A}_f))$-modules

$$\Xi_{\mathcal{P}}^{\mathcal{G}} \colon S(\check{\mathfrak{a}}_{\mathcal{P}}^{\mathcal{G}}) \otimes C^\infty(\mathcal{P}(\mathbb{A})\backslash\mathcal{G}(\mathbb{A})) \cong \operatorname{Ind}_{\mathcal{P}}^{\mathcal{G}} S(\check{\mathfrak{a}}_{\mathcal{P}}^{\mathcal{G}}) \otimes \mathbb{C}_{2\rho_{\mathcal{P}}} \to A_{\mathcal{J},\mathcal{I}} \tag{5.10}$$

which maps $D \otimes \phi$ to $\left(D q_{\mathcal{P}}^{\mathcal{G}}(\cdot)E_{\mathcal{P}}^{\mathcal{G}}(\phi, \cdot)\right)(\rho_{\mathcal{P}})$.

To see that the functions in the image of $\Xi_{\mathcal{P}}^{\mathcal{G}}$ are annihilated by sufficiently high powers of $\mathcal{I}_S$ and $\mathcal{J}$, it suffices to note that $\operatorname{Ind}_{\mathcal{P}}^{\mathcal{G}} S(\check{\mathfrak{a}}_{\mathcal{P}}^{\mathcal{G}}) \otimes \mathbb{C}_{2\rho_{\mathcal{P}}}$ is the union of an ascending sequence of subrepresentations with quotients isomorphic to $\operatorname{Ind}_{\mathcal{P}}^{\mathcal{G}} \mathbb{C}_{2\rho_{\mathcal{P}}}$, and that $\mathcal{I}_S$ and $\mathcal{J}$ trivially act on $\operatorname{Ind}_{\mathcal{P}}^{\mathcal{G}} \mathbb{C}_{2\rho_{\mathcal{P}}}$.

We first prove the surjectivity of $\Xi_{\mathcal{P}_o}^{\mathcal{G}}$.

Theorem 5.1. *$\Xi_{\mathcal{P}_o}^{\mathcal{G}}$ is surjective. It is independent of the choice of $\boldsymbol{K}_f$.*

Proof. The fact that Ξ is independent of $\boldsymbol{K}_f$ is established by an easy computation, using the fact that both $E_{\mathcal{P}_o}^{\mathcal{G}}$ and the identification

$$S(\check{\mathfrak{a}}_o^{\mathcal{G}}) \otimes C^\infty(\mathcal{P}_o(\mathbb{A})\backslash\mathcal{G}(\mathbb{A})) \cong \operatorname{Ind}_{\mathcal{P}_o}^{\mathcal{G}} S(\check{\mathfrak{a}}_o^{\mathcal{G}}) \otimes \mathbb{C}_{2\rho_o}$$

depend on $\boldsymbol{K}_f$, and these dependencies cancel out.

We will derive the surjectivity of $\Xi_{\mathcal{P}_o}^{\mathcal{G}}$ from the description of the space of automorphic forms in [11, §1]. Recall from [11, Theorem 1.4] that the space $A_{\mathcal{J}}$ as a composition

$$A_{\mathcal{J}} = \bigoplus_{\{P\}} \coprod_{\varphi \in \Phi_{\mathbb{C},\{P\}}} \mathfrak{A}_{\mathbb{C},\{P\},\varphi}, \tag{5.11}$$

where the first sum is over classes $\{P\}$ of associate parabolic subgroups and the second sum is over $\Phi_{\mathbb{C},\{P\}}$, a set of equivalence classes of cuspidal automorphic representations π of the Levi components of the elements of $\{P\}$. Here two cuspidal automorphic representations belong to the same equivalence class if they can be identified by a Weyl group substitution. An equivalence class belongs to $\Phi_{\mathbb{C},\{P\}}$ if and only if it is in a certain way compatible with the infinitesimal character $\mathcal{J}$ of the constant representation. For a precise definition, we refer to [11, §1.2]. Note that our notation is slightly different from the notation in [11], where the space of automorphic forms was denoted $\mathfrak{A}_{\mathcal{E}}$ with a finite-dimensional representation $\mathcal{E}$, which in our case is $\mathbb{C}$. Therefore,

$A_{\mathcal{J}}$ in our notations is $\mathfrak{A}_{\mathbb{C}}$ in [11]. The notations on the left side of (5.11) are, however, the same as in [11].

By [11, Theorem 1.4], the space $\mathfrak{A}_{\mathbb{C},\{P\},\varphi}$ can be spanned by the coefficients of the Laurent expansion of cuspidal Eisenstein series starting from elements of φ. In particular, [11, Theorem 1.4] says that, for the special case $\{P\} = \{\mathcal{P}_o\}$ and $\varphi = \{\mathbb{C}_{w\cdot\rho_o}\}_{w\in W(\mathcal{A}_o:\mathcal{G}(\mathbb{Q}))}$, we have

$$\text{image of } \Xi^{\mathcal{G}}_{\mathcal{P}_o} = \mathfrak{A}_{\mathbb{C},\{\mathcal{P}_o\},\{\mathbb{C}_{w\cdot\rho_o}\}_{w\in W(\mathcal{A}_o:\mathcal{G}(\mathbb{Q}))}}. \tag{5.12}$$

Let us fix $\{P\}$ and $\varphi \in \Phi_{\mathbb{C},\{P\}}$. Let $\mathcal{P} \in \{P\}$ and let π be an irreducible cuspidal automorphic representation of $\mathcal{L}_{\mathcal{P}}$ which belongs to $\varphi_{\mathcal{P}}$. Let $\chi_\pi\colon \mathcal{A}_{\mathcal{P}}(\mathbb{A})/\mathcal{A}_{\mathcal{P}}(\mathbb{Q}) \to \mathbb{C}^\times$ be the central character of π, and let $\lambda_\pi \in \check{\mathfrak{a}}^{\mathcal{G}}_{\mathcal{P}}$ be the differential of the restriction of χ_π to $\mathcal{A}_{\mathcal{P}}(\mathbb{R})$. By applying a Weyl group substitution to $\mathcal{P}$ and π, we may assume $\lambda_\pi \in \overline{\check{\mathfrak{a}}^{\mathcal{G}+}_{\mathcal{P}}}$. Let S be a set of non-archimedean primes of $\mathbb{Q}$ which has a finite complement. We assume that π is unramified at the places of S. Let $v \in S$. By [11, Theorem 2.3], we have an ideal $\mathcal{I}_{\varphi,v} \subset \mathfrak{H}_v$ associated to φ such that all $\boldsymbol{K}_v$-spherical vectors in $\mathfrak{A}_{\mathbb{C},\{P\},\varphi}$ are annihilated by some power of $\mathcal{I}_{\varphi,v}$. Recall the annihilator $\mathcal{I}_v \subset \mathfrak{H}_v$ of the constant representation. If $\mathfrak{A}_{\mathbb{C},\{P\},\varphi} \cap A_{\mathcal{J},\mathcal{I}} \neq \{0\}$, then we must have $\mathcal{I}_{\varphi,v} = \mathcal{I}_v$ for all but finitely many places. We will verify that this implies $\{P\} = \{\mathcal{P}_o\}$ and $\varphi = \{w \cdot \mathbb{C}_{2\rho_o}\}_{w\in W(\mathcal{A}_o:\mathcal{G}(\mathbb{Q}))}$. By (5.12), this will complete the proof of the theorem.

Let $v \in S$ such that $\mathcal{I}_{\varphi,v} = \mathcal{I}_v$. We recall Satake's description of $\mathfrak{H}_v$. Let $\mathcal{P}_v \subset \mathcal{P}_o$ be a minimal $\mathbb{Q}_v$-rational parabolic subgroup with Levi component $\mathcal{L}_v$. Let

$$\check{\mathfrak{a}}_v = X^*(\mathcal{P}_v)_{\mathcal{Q}_v} \otimes_{\mathbb{Z}} \mathbb{R},$$

where $X^*_{\mathbb{Q}_v}$ are the characters defined over $\mathbb{Q}_v$, and let $\mathfrak{a}_v$ be the dual of $\check{\mathfrak{a}}_v$. Let $\boldsymbol{T}_v$ be the group of unramified characters of $\mathcal{L}_v(\mathbb{Q}_v)$, i.e., of continuous characters $\chi\colon \mathcal{L}_v(\mathbb{Q}_v) \to \mathbb{C}^\times$ which are trivial on the projection of $\mathcal{P}_v(\mathbb{Q}_v)\cap\boldsymbol{K}_v$ to $\mathcal{L}_v(\mathbb{Q}_v)$. The map

$$\begin{aligned}(\check{\mathfrak{a}}_v)_{\mathbb{C}} &\to \boldsymbol{T}_v\\ \lambda &\to \chi_\lambda(l) = \mathrm{e}^{\langle H_{\mathcal{P}_v}(l),\chi\rangle}\end{aligned} \tag{5.13}$$

is surjective, and $\boldsymbol{T}_v$ has the structure of a complex torus which is isomorphic to $(\check{\mathfrak{a}}_v)_{\mathbb{C}}/\Gamma_v$, where Γ_v is a lattice in $i\check{\mathfrak{a}}_v$. Let $\mathcal{O}(T_v)$ be the ring of algebraic functions on the complex torus $\boldsymbol{T}_v$. The Weyl group $W(\mathcal{A}_v : \mathcal{G}(\mathbb{Q}_v))$ of $\mathcal{A}_v$ in $\mathcal{G}(\mathbb{Q}_v)$ acts on $\mathfrak{T}_v$, and we have the Satake isomorphisms

$$\boldsymbol{S}_{\mathcal{G}(\mathbb{Q}_v)}\colon \mathfrak{H}_v \to \mathcal{O}(\boldsymbol{T}_v)^{W\left(\mathcal{A}_v:\mathcal{G}(\mathbb{Q}_v)\right)}$$

$$\boldsymbol{S}_{\mathcal{L}_{\mathcal{P}}(\mathbb{Q}_v)}\colon \mathfrak{H}_v(\mathcal{L}_{\mathcal{P}}) \to \mathcal{O}(\boldsymbol{T}_v)^{W\left(\mathcal{A}_v:\mathcal{L}_{\mathcal{P}}(\mathbb{Q}_v)\right)}$$

(cf. [8, Theorem 4.1]) for $\mathcal{G}$ and for the Levi components of standard parabolic subgroups. Here $\mathfrak{H}_v(\mathcal{L}_{\mathcal{P}})$ is the Hecke algebra for $\mathcal{L}_{\mathcal{P}}(\mathbb{Q}_v)$, defined by the projection of $\boldsymbol{K}_v \cap \mathcal{P}(\mathbb{Q}_v)$ to $\mathcal{L}_{\mathcal{P}}(\mathbb{Q}_v)$.

Let $\mathfrak{H}_v(\mathcal{L}_\mathcal{P})$ act on the $\boldsymbol{K}_v$-spherical vector of π by multiplication by the character $(\boldsymbol{S}_{\mathcal{L}_\mathcal{P}(\mathbb{Q}_v)}h)(t_v)$ for $t_v \in \boldsymbol{T}_v$. The $W(\mathcal{A}_v : \mathcal{L}_\mathcal{P}(\mathbb{Q}_v))$-orbit of t_v is uniquely determined by π. Let $\tilde{t}_v \in \check{\mathfrak{a}}_v$ be a lifting of t_v. It is well-known that the ideal $\mathcal{I}_v$ corresponds to the image of ρ_v in $\boldsymbol{T}_v$ by (5.13), where ρ_v is one half the sum of the positive roots of $\mathcal{A}_v$. If $\mathcal{I}_v = \mathcal{I}_{\varphi,v}$, then the $W(\mathcal{A}_v : \mathcal{G}(\mathbb{Q}_v))$-orbit of that image must contain t_v. By changing $\tilde{t}_v$ in its Γ_v-orbit, we may assume

$$t_v = w\rho_v \tag{5.14}$$

for some $w \in W(\mathcal{A}_v : \mathcal{G}(\mathbb{Q}_v))$. Let $\check{\mathfrak{a}}_v^\mathcal{P}$ be defined in a similar way as $\check{\mathfrak{a}}_o^\mathcal{P}$, and let $t_v = t_v^\mathcal{P} + t_{v\mathcal{P}}$ be the decomposition of t_v according to $\check{\mathfrak{a}}_v = \check{\mathfrak{a}}_v^\mathcal{P} \oplus \check{\mathfrak{a}}_{\mathcal{P},v}$, where $\check{\mathfrak{a}}_{\mathcal{P},v} \supseteq \check{\mathfrak{a}}_\mathcal{P}$ is the $\mathbb{Q}_v$-character group of $\mathcal{P}$ made into a real vector space. By changing t_v in its $W(\mathcal{A}_v : \mathcal{L}_\mathcal{P}(\mathbb{Q}_v))$-orbit, we may assume that $t_v^\mathcal{P}$ belongs to the closure of the positive Weyl chamber $\overline{\check{\mathfrak{a}}_v^{\mathcal{P}+}}$.

Let Δ_v and $\Delta_v^\mathcal{P}$ be the same as in the proof of Proposition 4.3. For a root α of $\mathcal{A}_v$, let n_α be its multiplicity. If α is positive and reduced, then we have the inequality

$$\langle \check{\alpha}, \rho_v \rangle \geq n_\alpha + 2n_{2\alpha}, \tag{5.15}$$

for which equality occurs if and only if α is simple. This is easily verified by comparing the expressions

$$s_\alpha \rho_v = \rho_v - \alpha\langle \check{\alpha}, \rho_v \rangle = \rho_v - \sum_{\substack{\beta>0 \\ s_\alpha\beta<0}} n_\beta \beta.$$

From (5.14) and (5.15), we get for $\alpha \in \Delta_v^\mathcal{P}$

$$\begin{aligned} \left|\langle \check{\alpha}, t_v^\mathcal{P} \rangle\right| &= |\langle \check{\alpha}, t_v \rangle| \\ &\geq n_\alpha + n_{2\alpha} \\ &= \langle \check{\alpha}, \rho_v^\mathcal{P} \rangle. \end{aligned}$$

This implies $t_v^\mathcal{P} \in \rho_v^\mathcal{P} + \overline{\check{\mathfrak{a}}_v^{\mathcal{P}+}}$. By the boundedness of the matrix coefficients of the unitary representation π, this may happen only if $t_v^\pi = \rho_v^\mathcal{P}$. But then the local factor π_v of π at v is multiplication by an unramified character of $\mathcal{L}(\mathbb{Q}_v)$. Since this has to be the case at all but finitely many primes, weak approximation proves that π must be one-dimensional. Since π is cuspidal, this implies $\mathcal{P} = \mathcal{P}_o$.

To show that $\pi = \mathbb{C}_{\rho_o}$, it remains to verify that $t_{v\mathcal{P}_o} = \rho_o$. Fix a Weyl group invariant scalar product on $\check{\mathfrak{a}}_v$ and consider the following inequality:

$$\begin{aligned} \left|t_{v\mathcal{P}_o}\right|^2 &= \langle t_{v\mathcal{P}_o}, t_v \rangle \\ &\leq \langle t_{v\mathcal{P}_o}, \rho_v \rangle \\ &= \langle t_{v\mathcal{P}_o}, \rho_o \rangle \\ &\leq |t_{v\mathcal{P}_o}|\,|\rho_o|\,. \end{aligned} \tag{5.16}$$

The equalities are easy orthogonality relations. The inequality on the second line follows from $t_v = w\rho_v \in \rho_v - \overline{{}^{+}\check{\mathfrak{a}}_v}$, where $\overline{{}^{+}\check{\mathfrak{a}}_v}$ is the closed positive cone spanned by the positive roots, plus the fact that by our assumption on π we have $t_{v\mathcal{P}_o} = \lambda_\pi \in \overline{\check{\mathfrak{a}}_o^{\mathcal{G}+}}$ for the central character $\lambda + \pi$ of π. The inequality on the last line of (5.16) is the Cauchy-Schwarz inequality. We also have the equality

$$|t_{v\mathcal{P}_o}|^2 = |t_v|^2 - \left|t_v^{\mathcal{P}_o}\right|^2 = |\rho_v|^2 - \left|\rho_v^{\mathcal{P}_o}\right|^2 = |\rho_o|^2 .$$

Comparing this with (5.16), we see that equality must occur on the last line of (5.16). By Cauchy–Schwarz, this implies $t_{v\mathcal{P}_o} = \rho_o$, and we have finally verified that $\mathcal{P} = \mathcal{P}_o$ and $\pi = \mathbb{C}_{\rho_o}$. As was mentioned earlier, this completes the proof. □

Our next task is to determine the kernel of $\Xi_{\mathcal{P}_o}^{\mathcal{G}}$. We start with a few facts about the kernel of the operators $\tau_{\mathcal{P}_o}^{\mathcal{P}}$. The operator $\tau_{\mathcal{P}_o}^{\mathcal{G}}$ is a $\mathcal{G}(\mathbb{A})$-invariant linear functional on $\operatorname{Ind}_{\mathcal{P}}^{\mathcal{G}} \mathbb{C}_{2\rho_o}$ and induces a duality

$$\begin{aligned} C^\infty(\mathcal{P}_o(\mathbb{A})\backslash\mathcal{G}(\mathbb{A}) \otimes \operatorname{Ind}_{\mathcal{P}_o}^{\mathcal{G}} \mathbb{C}_{2\rho_o} &\to \mathbb{C} \\ \phi \otimes \tilde{\phi} &\to \tau_{\mathcal{P}_o}^{\mathcal{G}}(\phi\tilde{\phi}). \end{aligned}$$

With respect to this pairing, for any standard parabolic subgroup $\mathcal{P}$ with $\dim \mathfrak{a}_o^{\mathcal{P}} = 1$, the orthogonal complement of $C^\infty(\mathcal{P}(\mathbb{A})\backslash\mathcal{G}(\mathbb{A}))$ is $\operatorname{Ind}_{\mathcal{P}}^{\mathcal{G}} \check{\mathfrak{S}}t_{\mathcal{L}_{\mathcal{P}}(\mathbb{A})}$. For arbitrary $\mathcal{P} \neq \mathcal{P}_o$, the orthogonal complement of $C^\infty(\mathcal{P}(\mathbb{A})\backslash\mathcal{G}(\mathbb{A}))$ is the kernel of $\tau_{\mathcal{P}}$. By Theorem 4.2 applied to $\boldsymbol{C}(\mathcal{L}_{\mathcal{P}}, \mathcal{P}_o/\mathcal{N}_{\mathcal{P}}, \mathbb{A})^\bullet$, we have

$$C^\infty(\mathcal{P}(\mathbb{A})\backslash\mathcal{G}(\mathbb{A})) = \bigcap_{\substack{\mathcal{Q}\subset\mathcal{P} \\ \dim \mathfrak{a}_o^{\mathcal{Q}}=1}} C^\infty(\mathcal{Q}(\mathbb{A})\backslash\mathcal{G}(\mathbb{A})),$$

and the orthogonal complement of the intersection is the sum of the orthogonal complements since any $\mathbb{K}$-type occurs with finite multiplicity. We get

$$\ker \tau_{\mathcal{P}_o}^{\mathcal{P}} = \sum_{\substack{\mathcal{Q}\subset\mathcal{P} \\ \dim \mathfrak{a}_o^{\mathcal{Q}}=1}} \operatorname{Ind}_{\mathcal{Q}}^{\mathcal{G}} \check{\mathfrak{S}}t_{\mathcal{L}_{\mathcal{Q}}(\mathbb{A})}. \tag{5.17}$$

We will now give the description of the kernel of $\Xi_{\mathcal{P}_o}^{\mathcal{G}}$.

Theorem 5.2. *We have*

$$\ker \Xi_{\mathcal{P}_o}^{\mathcal{G}} = \sum_{\substack{\mathcal{P}\in\mathfrak{P} \\ \dim \mathfrak{a}_o^{\mathcal{P}}=1}} \operatorname{Ind}_{\mathcal{P}}^{\mathcal{G}} \check{\mathfrak{S}}t_{\mathcal{L}_{\mathcal{P}}(\mathbb{A})} \otimes S(\check{\mathfrak{a}}_{\mathcal{P}}^{\mathcal{G}}) \otimes \mathbb{C}_{2\rho_{\mathcal{P}}}. \tag{5.18}$$

Proof. It is clear from (5.6) that the right-hand side of (5.18) is really contained in the kernel of $\Xi_{\mathcal{P}_o}^{\mathcal{G}}$. Conversely, let $f \in \operatorname{Ind}_{\mathcal{P}_o}^{\mathcal{G}} S(\check{\mathfrak{a}}_o^{\mathcal{G}}) \otimes \mathbb{C}_{2\rho_o}$ belong to the kernel of $\Xi_{\mathcal{P}_o}^{\mathcal{G}}$. Define $\delta_{\mathcal{P}} \in S(\check{\mathfrak{a}}_{\mathcal{P}}^{\mathcal{G}})$ by

$$\delta_{\mathcal{P}} = \prod_{\alpha \in \Delta_{\mathcal{P}}} \omega_\alpha,$$

where ω_α is defined by

$$\langle \omega_\alpha, \check{\beta} \rangle = \begin{cases} \{0\} & \text{if } \beta \in \Delta_{\mathcal{P}} \setminus \{\alpha\} \\ 1 & \text{if } \alpha = \beta. \end{cases}$$

There is a unique decomposition

$$f = \sum_{\mathcal{P} \in \mathfrak{P}} f^{(\mathcal{P})} \delta_{\mathcal{P}}$$

with

$$f_{(\mathcal{P})} \in \operatorname{Ind}_{\mathcal{P}_o}^{\mathcal{G}} S(\check{\mathfrak{a}}_{\mathcal{P}}^{\mathcal{G}}) \otimes \mathbb{C}_{2\rho_o}.$$

Of course, the map $f \to f^{(\mathcal{P})}$ is only a map of vector spaces. From the fact that $f \in \ker \Xi_{\mathcal{P}_o}^{\mathcal{G}}$ we derive

$$(\mathrm{Id} \otimes \tau_{\mathcal{P}_o}^{\mathcal{P}}) f^{(\mathcal{P})} = 0 \in \operatorname{Ind}_{\mathcal{P}}^{\mathcal{G}} S(\check{\mathfrak{a}}_{\mathcal{P}}^{\mathcal{G}}) \otimes \mathbb{C}_{2\rho_{\mathcal{P}}}. \tag{5.19}$$

By (5.17) this implies

$$f^{(\mathcal{P})} \delta_{\mathcal{P}} \in \sum_{\substack{\mathcal{Q} \subseteq \mathcal{P} \\ \dim \mathfrak{a}_o^{\mathcal{Q}} = 1}} \operatorname{Ind}_{\mathcal{Q}}^{\mathcal{G}} S(\check{\mathfrak{a}}_{\mathcal{P}}^{\mathcal{G}}) \otimes \check{\mathfrak{St}}_{\mathcal{L}_{\mathcal{Q}}(\mathbb{A})} \otimes \mathbb{C}_{2\rho_{\mathcal{Q}}}$$

and proves (5.18).

Let T be a bijective map from the set of vectors $\rho_{\mathcal{P}}$ for $\mathcal{P} \in \mathfrak{P}$ to the set $\{0; 1; \ldots; 2^{\dim \mathfrak{a}_o^{\mathcal{G}}} - 1\}$ with the following property: If $\rho_{\mathcal{Q}} \in \rho_{\mathcal{P}} - \overline{{}^+\check{\mathfrak{a}}_o^{\mathcal{G}}}$, then $T(\rho_{\mathcal{P}}) \leq T(\rho_{\mathcal{Q}})$. Here $\overline{{}^+\check{\mathfrak{a}}_o^{\mathcal{G}}}$ is the closed cone spanned by Δ_o. It easy to verify the existence of such a function T. Let $\mathcal{P}^{(i)}$ be the unique parabolic subgroup with $T(\rho_{\mathcal{P}^{(i)}}) = i$. Then $\mathcal{P}^{(0)} = \mathcal{P}_o$.

It is a consequence of (5.6) that

$$\Xi_{\mathcal{P}_o}^{\mathcal{G}} f = \sum_{\mathcal{P}} \Xi_{\mathcal{P}}^{\mathcal{G}} \left((\mathrm{Id} \otimes \tau_{\mathcal{P}_o}^{\mathcal{P}}) f^{(\mathcal{P})} \delta_{\mathcal{P}} \right). \tag{5.20}$$

We will prove (5.19) for $\mathcal{P} = \mathcal{P}^{(i)}$ by induction on i by an investigation of the constant term of the Eisenstein series occurring in (5.20). Recall that for a continuous function ψ on $\mathcal{G}(\mathbb{Q}) \backslash \mathcal{G}(\mathbb{A})$, the constant term with respect to $\mathcal{P}$ is defined by

$$\psi_{\mathcal{P}}(g) = \int_{\mathcal{N}_{\mathcal{P}}(\mathbb{Q}) \backslash \mathcal{N}_{\mathcal{G}}(\mathbb{A})} \psi(ng)\, dn,$$

where the Haar measure dn is normalised by $1_{\mathcal{P}} = 1$. The necessary facts about the constant term of Eisenstein series are summarised in the following lemma, which will be proved after the proof of Theorem 5.2 is complete.

Lemma 5.3. *There exists a finite set $\mathfrak{W}_i$ of affine maps $\check{\mathfrak{a}}_{\mathcal{P}^{(i)}} \to \check{\mathfrak{a}}_o$ such that*

$$\left(E^{\mathcal{G}}_{\mathcal{P}^{(i)}}(\phi,\lambda)\right)_{\mathcal{P}_o}(g) = \sum_{w\in\mathfrak{W}_i} \left(N_i(w,\lambda)\phi\right)(g)\,\mathrm{e}^{\langle w\lambda+\rho_o, H_{\mathcal{P}_o}(g)\rangle}, \tag{5.21}$$

where $N_i(w,\lambda)$ is a meromorphic function from $\check{\mathfrak{a}}_{\mathcal{P}}$ to the space of $\mathbb{K}$-invariant homomorphisms from $C^\infty(\mathcal{P}(\mathbb{A})\backslash\mathcal{G}(\mathbb{A}))$ to $C^\infty(\mathcal{P}_o(\mathbb{A})\backslash\mathcal{G}(\mathbb{A}))$. If w_i is defined by

$$\begin{aligned} w_i\colon \check{\mathfrak{a}}_{\mathcal{P}^{(i)}} &\to \check{\mathfrak{a}}_o \\ w_i\lambda &= \lambda - \rho_o^{\mathcal{P}^{(i)}}, \end{aligned}$$

then $w_i \in \mathfrak{W}_i$ and $N_i(w_i,\lambda)\phi = \phi$. Furthermore, if $w \in \mathfrak{W}_j$ and if $w\rho_{\mathcal{P}^{(j)}} = \rho_{\mathcal{P}^{(i)}} - \rho_o^{\mathcal{P}^{(i)}}$, then $j \le i$.

Let us assume that (5.19) has been proved for $\mathcal{P} = \mathcal{P}^{(j)}$ with $j < i$. If $i = 0$, this assumption is void. In any case, the induction assumption implies that the only summands in (5.20) which are possibly different from zero belong to the parabolic subgroups $\mathcal{P}^{(j)}$ with $j \ge i$. As a consequence of (5.21), the constant term of $\Xi^{\mathcal{G}}_{\mathcal{P}_o} f$ may be written as

$$\left(\Xi^{\mathcal{G}}_{\mathcal{P}_o} f\right)_{\mathcal{P}_o}(g) = \sum_{\lambda\in\Lambda} f_\lambda(g)\,\mathrm{e}^{\langle\lambda+\rho_o, H_{\mathcal{P}_o}(g)\rangle},$$

where Λ is a finite subset of $\check{\mathfrak{a}}_o$ and where f_λ is a continuous function on $\mathcal{G}(\mathbb{A})$ with the property that for any $g \in \mathcal{G}(\mathbb{A})$, the function $f_\lambda(pg)$ of $p \in \mathcal{P}_o(\mathbb{A})$ is a polynomial in $H_{\mathcal{P}_o}(p)$. Since f is in the kernel of $\Xi^{\mathcal{G}}_{\mathcal{P}_o}$, we have $f_\lambda = 0$ for any λ.

Let $N = \dim \mathfrak{a}^{\mathcal{G}}_{\mathcal{P}^{(i)}}$, let $\alpha_1,\ldots,\alpha_N$ be the elements of $\Delta_{\mathcal{P}^{(i)}}$, and let $\omega_i = \omega_{\alpha_i}$. We have a unique representation

$$(\mathrm{Id}\otimes\tau^{\mathcal{P}}_{\mathcal{P}_o}) f^{(\mathcal{P})} = \sum_{a_1,\ldots,a_N=0}^{\infty} \left(\prod_{k=1}^{N} \omega_i^{a_i}\right) \otimes f_{a_1,\ldots,a_N}$$

with $f_{a_1,\ldots,a_N} \in \mathrm{Ind}^{\mathcal{G}}_{\mathcal{P}^{(i)}} \mathbb{C}_{2\rho_{\mathcal{P}^{(i)}}}$. By the induction assumption, (5.20), the definition of $\Xi^{\mathcal{G}}_{\mathcal{P}}$ and Lemma 5.3, we have

$$f_{\rho_{\mathcal{P}^{(i)}}-\rho_o^{\mathcal{P}^{(i)}}}(g) = \sum_{a_1,\ldots,a_N=0}^{\infty} \left(\prod_{k=1}^{N} (a_i+1)\langle\omega_i, H_{\mathcal{P}_o}(g)\rangle^{a_i}\right) f_{a_1,\ldots,a_N}(g).$$

This function vanishes identically if and only if $f_{a_1,\ldots,a_N} = 0$ for all choices of the a_i. This establishes (5.19) and completes the proof of the theorem. □

Proof (of Lemma 5.3). The formula (5.21) is a general fact from the theory of Eisenstein systems (cf. [15, §7] or the modern exposition [18, §IV]). In general, the theory of Eisenstein systems provides for the possibility of additional polynomial factors of higher degree in the expression for the constant term. Since this may happen only in the case of singular infinitesimal character, in our case the expression for the constant term simplifies to (5.21).

To arrive at the assertion about $N_i(w_i, \lambda)$, we consider the partial Eisenstein series $E_{\mathcal{P}}^{\mathcal{R}}(\phi, \lambda)$, which is defined as in (5.5), but with the summation restricted to $\mathcal{P}(\mathbb{Q})\backslash\mathcal{R}(\mathbb{Q})$. As a general fact about Eisenstein systems, the constant term of $E_{\mathcal{P}}^{\mathcal{R}}(\phi, \lambda)$ is given as in (5.21), but with the summation restricted to those $w \in \mathfrak{W}_i$ whose linear part is the identity on $\check{\mathfrak{a}}_{\mathcal{R}}$. In the special case $\mathcal{R} = \mathcal{P}$, where

$$E_{\mathcal{P}}^{\mathcal{P}}(\phi, \lambda) = \mathrm{e}^{\langle \lambda + \rho_{\mathcal{P}}, H_{\mathcal{P}}(\cdot) \rangle}\, \phi(\cdot),$$

this expression for the constant term boils down to the assertion about $N_i(w_i, \lambda)$.

Finally, the only Eisenstein series $E_{\mathcal{P}}^{\mathcal{G}}(\phi, \rho_{\mathcal{P}})$ which have an exponential term of the form $\mathrm{e}^{\langle 2\rho_{\mathcal{P}^{(i)}}, H_{\mathcal{P}^{(i)}} \rangle}$ in their constant terms are the Eisenstein series starting from $\mathcal{P} = \mathcal{P}^{(j)}$ with $j \leq i$. This fact is a consequence of our condition on T and the proof of the main theorem in [9, §6]. The proof of Lemma 5.3 is complete. □

The description of the kernel of $\Xi_{\mathcal{P}_o}^{\mathcal{G}}$ is a little too complicated to use directly. Therefore, we will use it to get a resolution of the space of automorphic forms by induced representations whose cohomology can be described easily. This is achieved in two steps. In the first step, we consider the functor

$$\boldsymbol{F}^{\mathcal{P}} = \left\{ \begin{array}{ll} \mathrm{Ind}_{\mathcal{P}}^{\mathcal{G}} S(\check{\mathfrak{a}}_{\mathcal{P}}^{\mathcal{G}}) \otimes \check{\mathfrak{S}}t_{\mathcal{L}_{\mathcal{P}}(\mathbb{A})} \otimes \mathbb{C}_{2\rho_{\mathcal{P}}} & \text{if } \mathcal{P} \neq \mathcal{P}_o \\ \mathrm{Ind}_{\mathcal{P}_o}^{\mathcal{G}} S(\check{\mathfrak{a}}_o^{\mathcal{G}}) \otimes \mathbb{C}_{2\rho_o} & \text{if } \mathcal{P} = \mathcal{P}_o. \end{array} \right\} \subseteq \mathrm{Ind}_{\mathcal{P}_o}^{\mathcal{G}} S(\check{\mathfrak{a}}_o^{\mathcal{G}}) \otimes \mathbb{C}_{2\rho_o}.$$

The map $\boldsymbol{F}(\mathcal{G})^{\tilde{\mathcal{P}} \supseteq \mathcal{P}}$ is given by the inclusion $S(\check{\mathfrak{a}}_{\tilde{\mathcal{P}}}^{\mathcal{G}}) \subset S(\check{\mathfrak{a}}_{\mathcal{P}})$, followed by the inclusion

$$\check{\mathfrak{S}}t_{\mathcal{L}_{\tilde{\mathcal{P}}}(\mathbb{A})} \subseteq \mathrm{Ind}_{\mathcal{P}}^{\tilde{\mathcal{P}}} \check{\mathfrak{S}}t_{\mathcal{L}_{\mathcal{P}}(\mathbb{A})} \otimes \mathbb{C}_{2\rho_{\mathcal{P}}^{\tilde{\mathcal{P}}}}$$

which holds because of the description of $\check{\mathfrak{S}}t_{\mathcal{G}(\mathbb{A})}$ as the orthogonal complement of

$$\sum_{\mathcal{P} \supset \mathcal{P}_o} C^{\infty}(\mathcal{P}(\mathbb{A})\backslash\mathcal{G}(\mathbb{A})).$$

Proposition 5.4. *The map $\Xi_{\mathcal{P}_o}^{\mathcal{G}}$ defines an isomorphism*

$$H^{\dim \mathfrak{a}_o^{\mathcal{G}}}\left(C^*\left(\boldsymbol{F}(\mathcal{G})^{\bullet}\right)\right) \cong A_{\mathcal{J},\mathcal{I}}.$$

This is the only nonvanishing cohomology group of $C^*\left(\boldsymbol{F}(\mathcal{G})^{\bullet}\right)$.

Proof. Let the functor $\tilde{\boldsymbol{F}}^{\bullet}$ be defined by $\tilde{\boldsymbol{F}}^{\mathcal{P}} = \boldsymbol{F}(\mathcal{G})^{\mathcal{P}}$ if $\mathcal{P} \supset \mathcal{P}_o$ and

$$\tilde{\boldsymbol{F}}^{\mathcal{P}_o} = \sum_{\substack{\mathcal{P} \in \mathfrak{P} \\ \dim \mathfrak{a}_o^{\mathcal{P}} = 1}} \mathrm{Ind}_{\mathcal{P}}^{\mathcal{G}} \check{\mathfrak{S}}t_{\mathcal{L}_{\mathcal{P}}(\mathbb{A})} \otimes S(\check{\mathfrak{a}}_{\mathcal{P}}^{\mathcal{G}})\mathbb{C}_{2\rho_{\mathcal{P}}}.$$

This is our expression for the kernel of $\Xi^{\mathcal{G}}_{\mathcal{P}_o}$. It is therefore sufficient to prove the acyclicity of the chain complex of $\tilde{\boldsymbol{F}}^{\bullet}$.

We have a filtration of functors

$$\operatorname{Fil}_k \tilde{\boldsymbol{F}}^{\mathcal{P}} = \begin{cases} \sum\limits_{\substack{\mathcal{Q}\supseteq\mathcal{P} \\ \dim \mathfrak{a}^{\mathcal{G}}_{\mathcal{Q}}=k}} \operatorname{Ind}^{\mathcal{G}}_{\mathcal{P}} S(\check{\mathfrak{a}}^{\mathcal{G}}_{\mathcal{Q}}) \otimes \check{\mathfrak{St}}_{\mathcal{L}_{\mathcal{P}}(\mathbb{A})} \otimes \mathbb{C}_{2\rho_{\mathcal{P}}} & \text{if } \mathcal{P} \supset \mathcal{P}_o \\ \sum\limits_{\substack{\mathcal{Q}\in\mathfrak{P} \\ \dim \mathfrak{a}^{\mathcal{G}}_{\mathcal{Q}}=k}} \operatorname{Ind}^{\mathcal{G}}_{\mathcal{P}} S(\check{\mathfrak{a}}^{\mathcal{G}}_{\mathcal{Q}}) \otimes \check{\mathfrak{St}}_{\mathcal{L}_{\mathcal{P}}(\mathbb{A})} \otimes \mathbb{C}_{2\rho_{\mathcal{P}}} & \text{if } \mathcal{P} = \mathcal{P}_o \end{cases}$$

with quotients

$$(\operatorname{Fil}_k / \operatorname{Fil}_{k-1})\, \tilde{\boldsymbol{F}}^{\bullet} = \sum_{\substack{\mathcal{R}\in\mathfrak{P} \\ \dim \mathfrak{a}^{\mathcal{G}}_{\mathcal{R}}=k}} \boldsymbol{M}(\mathcal{R})^{\bullet},$$

where

$$\boldsymbol{M}(\mathcal{R})^{\mathcal{P}} = \begin{cases} 0 & \text{if } \mathcal{Q} \not\subseteq \mathcal{R} \\ S(\check{\mathfrak{a}}^{\mathcal{G}}_{\mathcal{R}}) \otimes \operatorname{Ind}^{\mathcal{G}}_{\mathcal{R}} \boldsymbol{D}(\mathcal{L}_{\mathcal{R}})^{\mathcal{P}/\mathcal{N}_{\mathcal{R}}} & \text{if } \mathcal{Q} \subseteq \mathcal{R}. \end{cases}$$

The acyclicity of the functors $\boldsymbol{D}(\mathcal{L}_{\mathcal{R}})^{\bullet}$ is the assertion of Theorem 4.5. This implies the acyclicity of the quotients of the filtration of $\tilde{\boldsymbol{F}}^{\bullet}$, and hence of $\tilde{\boldsymbol{F}}^{\bullet}$ itself. □

If $\mathcal{P} \supset \mathcal{P}_o$, then the cohomology of the representation $\boldsymbol{F}(\mathcal{G})^{\mathcal{P}}$ is still rather mysterious. We construct a second resolution for $A_{\mathcal{J},\mathcal{I}}$ by the bifunctor

$$\boldsymbol{G}(\mathcal{G})^{\mathcal{P}}_{\mathcal{Q}} = \begin{cases} \operatorname{Ind}^{\mathcal{G}}_{\mathcal{Q}} S(\check{\mathfrak{a}}^{\mathcal{G}}_{\mathcal{P}}) \otimes \mathbb{C}_{2\rho_{\mathcal{Q}}} & \text{if } \mathcal{Q} \subseteq \mathcal{P} \\ \{0\} & \text{if } \mathcal{Q} \not\subseteq \mathcal{P}. \end{cases}$$

The map $\boldsymbol{G}(\mathcal{G})^{\mathcal{P}}_{\tilde{\mathcal{Q}}\subseteq\mathcal{Q}}$ is given by $\tau^{\mathcal{Q}}_{\tilde{\mathcal{Q}}}$, and the map $\boldsymbol{G}(\mathcal{G})^{\mathcal{P}\subseteq\tilde{\mathcal{P}}}_{\mathcal{Q}}$ is given by the inclusion $S(\check{\mathfrak{a}}^{\mathcal{G}}_{\tilde{\mathcal{P}}}) \subseteq S(\check{\mathfrak{a}}^{\mathcal{G}}_{\mathcal{P}})$.

Proposition 5.5. *The map*

$$\operatorname{Ind}^{\mathcal{G}}_{\mathcal{P}_o} S(\check{\mathfrak{a}}^{\mathcal{G}}_{\mathcal{P}_o}) \otimes \mathbb{C}_{2\rho_o} = \boldsymbol{G}(\mathcal{G})^{\mathcal{P}_o}_{\mathcal{P}_o} \subset \mathcal{Z}^{\dim \mathfrak{a}^{\mathcal{G}}_o}\left(\boldsymbol{G}^{\bullet}_{\bullet}(\mathcal{G})\right)$$

induces a surjection

$$\operatorname{Ind}^{\mathcal{G}}_{\mathcal{P}_o} S(\check{\mathfrak{a}}^{\mathcal{G}}_{\mathcal{P}_o}) \otimes \mathbb{C}_{2\rho_o} \to H^{\dim \mathfrak{a}^{\mathcal{G}}_o}\left(C^*\left(\boldsymbol{G}(\mathcal{G})^{\bullet}_{\bullet}\right)\right)$$

whose kernel is equal to the kernel of $\Xi^{\mathcal{G}}_{\mathcal{P}_o}$. This gives us an isomorphism

$$H^{\dim \mathfrak{a}^{\mathcal{G}}_o}\left(C^*\left(\boldsymbol{G}(\mathcal{G})^{\bullet}_{\bullet}\right)\right) \cong A_{\mathcal{J},\mathcal{I}}.$$

The other cohomology groups of $C^*\left(\boldsymbol{G}(\mathcal{G})^{\bullet}_{\bullet}\right)$ vanish.

Proof. It suffices to construct an isomorphism

$$H^l\left(C^*\left(\boldsymbol{G}(\mathcal{G})^{\mathcal{P}}_{\bullet}\right)\right) = \begin{cases} \boldsymbol{F}(\mathcal{G})^{\mathcal{P}} & \text{if } l = 0 \\ \{0\} & \text{if } l > 0 \end{cases} \tag{5.22}$$

which is functorial in $\mathcal{P}$. Let us fix $\mathcal{P}$. Then

$$\boldsymbol{G}(\mathcal{G})^{\mathcal{P}}_{\bullet} = \operatorname{Ind}^{\mathcal{G}}_{\mathcal{P}} S(\check{\mathfrak{a}}^{\mathcal{G}}_{\mathcal{P}}) \otimes \boldsymbol{M}_{\bullet} \otimes \mathbb{C}_{2\rho_{\mathcal{P}}},$$

where

$$\boldsymbol{M}_{\mathcal{Q}} = \begin{cases} \operatorname{Ind}^{\mathcal{P}}_{\mathcal{Q}} \mathbb{C}_{2\rho^{\mathcal{P}}_{\mathcal{Q}}} & \text{if } \mathcal{Q} \subseteq \mathcal{P} \\ \{0\} & \text{if } \mathcal{Q} \not\subseteq \mathcal{P}. \end{cases}$$

If $\mathcal{Q} \subseteq \mathcal{P}$, then $\boldsymbol{M}_{\mathcal{Q}}$ is in duality with $\boldsymbol{C}\left(\mathcal{L}_{\mathcal{P}}, (\mathcal{P}_o/\mathcal{N}_{\mathcal{P}}), \mathbb{A}\right)$. An isomorphism (5.22) is therefore given by Theorem 4.2. It is easy to see that this isomorphism is functorial in $\mathcal{P}$. □

6 Construction of the isomorphism of equation (3.4)

Our final goal is to compute the $(\mathfrak{g}, \boldsymbol{K})$-hypercohomology of the chain complex $C^*(\boldsymbol{G}(\mathcal{G})^{\bullet}_{\bullet})$ and to relate it to the topological model explained in Section 3. We first compute $H^*_{(\mathfrak{m}_{\mathcal{G}}, \boldsymbol{K}^o_\infty)}\left(C^*\left(\boldsymbol{G}(\mathcal{G})^{\bullet}_{\mathcal{Q}}\right)\right)$ for a given parabolic subgroup $\mathcal{Q}$.

We have the projection

$$\begin{aligned} C^*_{(\mathfrak{m}_{\mathcal{G}}, \boldsymbol{K}^o_\infty)}\left(C^*\left(\boldsymbol{G}(\mathcal{G})^{\bullet}_{\mathcal{Q}}\right)\right) &\to C^*_{(\mathfrak{m}_{\mathcal{G}}, \boldsymbol{K}^o_\infty)}\left(\boldsymbol{G}(\mathcal{G})^{\mathcal{G}}_{\mathcal{Q}}\right) \\ &= C^*_{(\mathfrak{m}_{\mathcal{G}}, \boldsymbol{K}^o_\infty)}\left(\operatorname{Ind}^{\mathcal{G}}_{\mathcal{Q}} \mathbb{C}_{2\rho_{\mathcal{Q}}}\right). \end{aligned} \tag{6.1}$$

By Frobenius reciprocity we have

$$\begin{aligned} &C^*_{(\mathfrak{m}_{\mathcal{G}}, \boldsymbol{K}^o_\infty)}\left(\operatorname{Ind}^{\mathcal{G}}_{\mathcal{Q}} \mathbb{C}_{2\rho_{\mathcal{Q}}}\right) \cong \\ &\quad \left(\operatorname{Ind}^{\mathcal{G}(\mathbb{A}_f)}_{\mathcal{Q}(\mathbb{A}_f)} \mathbb{C}_{2\rho_{\mathcal{Q}}}\right) \otimes \left(\operatorname{Hom}_{\boldsymbol{K}^o_\infty \cap \mathcal{Q}(\mathbb{R})}\left(\Lambda^*(\mathfrak{q} \cap \mathfrak{m}_{\mathcal{G}}/\mathfrak{q} \cap \mathfrak{k}), \mathbb{C}\right)\right), \end{aligned} \tag{6.2}$$

where the $\mathcal{G}(\mathbb{A}_f)$-action on the second factor is trivial. The second factor carries the differential of the standard complex for computing $(\mathfrak{q} \cap \mathfrak{m}_{\mathcal{G}}, \boldsymbol{K}^o_\infty \cap \mathcal{Q}(\mathbb{R}))$-cohomology. The embedding

$$\det(\mathfrak{a}^{\mathcal{G}}_{\mathcal{Q}} \oplus \mathfrak{n}_{\mathcal{Q}}) \otimes \Lambda^*(\mathfrak{m}_{\mathcal{Q}}/\mathfrak{m}_{\mathcal{Q}} \cap \mathfrak{k}) \subset \Lambda^*(\mathfrak{q} \cap \mathfrak{m}_{\mathcal{G}}/\mathfrak{q} \cap \mathfrak{k})[\dim \mathfrak{a}^{\mathcal{G}}_{\mathcal{Q}} + \dim \mathfrak{n}_{\mathcal{Q}}]$$

defines a projection

$$\begin{aligned} p_{\mathcal{Q}}\colon\ &\operatorname{Hom}_{\boldsymbol{K}^o_\infty \cap \mathcal{Q}(\mathbb{R})}\left(\Lambda^*(\mathfrak{q} \cap \mathfrak{m}_{\mathcal{G}}/\mathfrak{q} \cap \mathfrak{k}), \mathbb{C}\right) \\ \to\ &\operatorname{Hom}_{\boldsymbol{K}^o_\infty \cap \mathcal{Q}(\mathbb{R})}\left(\Lambda^*(\mathfrak{m}_{\mathcal{Q}}/\mathfrak{m}_{\mathcal{Q}} \cap \mathfrak{k}) \otimes \det(\mathfrak{a}^{\mathcal{G}}_{\mathcal{Q}} \oplus \mathfrak{n}_{\mathcal{Q}}), \mathbb{C}\right)[-\dim \mathfrak{a}^{\mathcal{G}}_{\mathcal{Q}} - \dim \mathfrak{n}_{\mathcal{Q}}]. \end{aligned} \tag{6.3}$$

This is a homomorphism of chain complexes, and the differential of its target vanishes. Let $\boldsymbol{H}(\mathcal{G})_{\mathcal{Q}}{}^*$ be the graded vector space

$$\boldsymbol{H}(\mathcal{G})_{\mathcal{Q}}{}^* = \left(\operatorname{Ind}_{\mathcal{Q}(\mathbb{A}_f)}^{\mathcal{G}(\mathbb{A}_f)} \mathbb{C}_{2\rho_{\mathcal{Q}}}\right) \tag{6.4}$$
$$\otimes \operatorname{Hom}_{\boldsymbol{K}_\infty^\circ \cap \mathcal{Q}(\mathbb{R})} \left(\Lambda^*(\mathfrak{m}_{\mathcal{Q}}/\mathfrak{m}_{\mathcal{Q}} \cap \mathfrak{k}) \otimes \det(\mathfrak{a}_{\mathcal{Q}}^{\mathcal{G}} \oplus \mathfrak{n}_{\mathcal{Q}}), \mathbb{C}\right) [-\dim \mathfrak{a}_{\mathcal{Q}}^{\mathcal{G}} - \dim \mathfrak{n}_{\mathcal{Q}}],$$

which can also be viewed as a chain complex with zero differential. The composition of (6.1), (6.2), and (6.3) defines a projection

$$C^*_{(\mathfrak{m}_{\mathcal{G}}, \boldsymbol{K}_\infty^\circ)} \left(C^* \left(\boldsymbol{G}(\mathcal{G})^\bullet_{\mathcal{Q}}\right)\right) \to \boldsymbol{H}(\mathcal{G})_{\mathcal{Q}}{}^*. \tag{6.5}$$

Proposition 6.1. *The projection (6.5) defines an isomorphism on cohomology.*

Proof. By Frobenius reciprocity and by Kostant's theorem on $\mathfrak{n}$-homology ([27, Theorem 9.6.2] or [26, Theorem 3.2.3]), there is an isomorphism

$$\begin{aligned}
&H^*_{(\mathfrak{m}_{\mathcal{G}}, \boldsymbol{K}_\infty^\circ)} \left(\operatorname{Ind}_{\mathcal{Q}}^{\mathcal{G}} S(\check{\mathfrak{a}}_{\mathcal{P}}^{\mathcal{G}}) \otimes \mathbb{C}_{2\rho_{\mathcal{Q}}}\right) [\dim \mathfrak{n}_{\mathcal{Q}}] \\
&\cong \operatorname{Ind}_{\mathcal{Q}(\mathbb{A}_f)}^{\mathcal{G}(\mathbb{A}_f)} \left(\left\{H^*_{(\mathfrak{m}_{\mathcal{Q}}, \boldsymbol{K}_\infty^\circ \cap \mathcal{Q}(\mathbb{R}))}(\mathbb{C}) \otimes H^*_{\mathfrak{a}_{\mathcal{Q}}^{\mathcal{G}}} \left(S(\check{\mathfrak{a}}_{\mathcal{P}}^{\mathcal{G}})\right) \otimes \det \mathfrak{n}_{\mathcal{Q}}^{-1}\right\} \otimes \mathbb{C}_{2\rho_{\mathcal{Q}}}\right) \\
&\cong \operatorname{Ind}_{\mathcal{Q}(\mathbb{A}_f)}^{\mathcal{G}(\mathbb{A}_f)} \left(\left\{H^*_{(\mathfrak{m}_{\mathcal{Q}}, \boldsymbol{K}_\infty^\circ \cap \mathcal{Q}(\mathbb{R}))}(\mathbb{C}) \otimes \Lambda^*(\check{\mathfrak{a}}_{\mathcal{Q}}^{\mathcal{P}}) \otimes \det \mathfrak{n}_{\mathcal{Q}}^{-1}\right\} \otimes \mathbb{C}_{2\rho_{\mathcal{Q}}}\right) \qquad (6.6) \\
&\cong \operatorname{Ind}_{\mathcal{Q}(\mathbb{A}_f)}^{\mathcal{G}(\mathbb{A}_f)} \left(\left\{H^*_{(\mathfrak{m}_{\mathcal{Q}}, \boldsymbol{K}_\infty^\circ \cap \mathcal{Q}(\mathbb{R}))}(\mathbb{C}) \otimes \boldsymbol{E}(\mathcal{Q})^{\mathcal{P}^*} \otimes \det \mathfrak{n}_{\mathcal{Q}}^{-1}\right\} \otimes \mathbb{C}_{2\rho_{\mathcal{Q}}}\right),
\end{aligned}$$

where the factors in curved braces have trivial $\mathcal{Q}(\mathbb{A}_f)$-action. We have used the following isomorphism, which is easily constructed:

$$H^*_{\mathfrak{a}_{\mathcal{Q}}^{\mathcal{G}}} \left(S(\check{\mathfrak{a}}_{\mathcal{P}}^{\mathcal{G}})\right) \cong \Lambda^*(\check{\mathfrak{a}}_{\mathcal{Q}}^{\mathcal{P}}) \cong \boldsymbol{E}(\mathcal{Q})^{\mathcal{P}^*},$$

where $\boldsymbol{E}(\mathcal{Q})^{\mathcal{P}} = \Lambda^*(\check{\mathfrak{a}}_{\mathcal{Q}}^{\mathcal{P}})$ was considered at the end of Section 4. This isomorphism, and hence also (6.6), is functorial with respect to $\mathcal{P}$. (Recall that $\boldsymbol{E}(\mathcal{Q})^{\tilde{\mathcal{P}} \subseteq \mathcal{P}^*}$ is defined by the projection $\check{\mathfrak{a}}_{\mathcal{P}}^{\mathcal{Q}} \to \check{\mathfrak{a}}_{\tilde{\mathcal{P}}}^{\mathcal{Q}}$.)

If $\mathcal{P} = \mathcal{G}$, then the composition of the isomorphism (6.6) with the projection

$$\boldsymbol{E}(\mathcal{Q})^{\mathcal{G}} = \Lambda^*(\check{\mathfrak{a}}_{\mathcal{Q}}^{\mathcal{G}}) \to \det(\mathfrak{a}_{\mathcal{Q}}^{\mathcal{G}})^{-1}[-\dim \mathfrak{a}_{\mathcal{Q}}^{\mathcal{G}}] \tag{6.7}$$

is precisely the map defined by (6.2) and (6.3) on cohomology. By Lemma 4.6, the projection (6.7) defines an isomorphism

$$\begin{aligned}
&H^* \left(C^* \left(\operatorname{Ind}_{\mathcal{Q}(\mathbb{A}_f)}^{\mathcal{G}(\mathbb{A}_f)} \left(\left\{H^*_{(\mathfrak{m}_{\mathcal{Q}}, \boldsymbol{K}_\infty^\circ \cap \mathcal{Q}(\mathbb{R}))}(\mathbb{C}) \otimes \boldsymbol{E}(\mathcal{Q})^{\mathcal{P}^*} \otimes \det \mathfrak{n}_{\mathcal{Q}}^{-1}\right\} \otimes \mathbb{C}_{2\rho_{\mathcal{Q}}}\right)\right)\right) \\
&\cong \operatorname{Ind}_{\mathcal{Q}(\mathbb{A}_f)}^{\mathcal{G}(\mathbb{A}_f)} \left(\left\{H^*_{(\mathfrak{m}_{\mathcal{Q}}, \boldsymbol{K}_\infty^\circ \cap \mathcal{Q}(\mathbb{R}))}(\mathbb{C}) \otimes \boldsymbol{E}(\mathcal{Q})^{\mathcal{P}^*} \otimes \det(\mathfrak{a}_{\mathcal{Q}}^{\mathcal{G}} \oplus \mathfrak{n}_{\mathcal{Q}})^{-1}\right\} \otimes \mathbb{C}_{2\rho_{\mathcal{Q}}}\right) \\
&\qquad [-\dim \mathfrak{a}_{\mathcal{Q}}^{\mathcal{G}}],
\end{aligned}$$

which proves our claim. □

We now have to determine the structure of a covariant functor on $\boldsymbol{H}(\mathcal{G})_{\mathcal{Q}}{}^{*}$ such that (6.5) becomes functorial in $\mathcal{Q}$. We have to introduce some new notation. For any $\mathcal{Q}$, let the Haar measure on $\boldsymbol{K}_\infty \cap \mathcal{Q}(\mathbb{R})$ be normalized by $\int_{\boldsymbol{K}_\infty \cap \mathcal{Q}(\mathbb{R})} dk = 1$. Then there is a unique homomorphism

$$\tau_{\mathcal{P}(\mathbb{A}_f)}^{\mathcal{Q}(\mathbb{A}_f)} \colon \operatorname{Ind}_{\mathcal{P}(\mathbb{A}_f)}^{\mathcal{G}(\mathbb{A}_f)} \mathbb{C}_{2\rho_{\mathcal{P}}} \to \operatorname{Ind}_{\mathcal{Q}(\mathbb{A}_f)}^{\mathcal{G}(\mathbb{A}_f)} \mathbb{C}_{2\rho_{\mathcal{P}}}$$

such that we have, for the standard model of the induced representation in the space of functions on the adelic group,

$$(\tau_{\mathcal{P}}^{\mathcal{Q}} f)(g_f g_\infty) = \tau_{\mathcal{P}(\mathbb{A}_f)}^{\mathcal{Q}(\mathbb{A}_f)} \, \mathrm{e}^{\langle H_{\mathcal{Q}}(g_\infty), 2\rho_{\mathcal{Q}} \rangle} \int_{\boldsymbol{K}_\infty \cap \mathcal{Q}(\mathbb{R})} f(g_f k k_\infty) \, dk, \tag{6.8}$$

where $g_f \in \mathcal{G}(\mathbb{A}_f)$ and $g_\infty = p_\infty k_\infty \in \mathcal{G}(\mathbb{R})$ with $p_\infty \in \mathcal{P}(\mathbb{R})$ and $k_\infty \in \boldsymbol{K}_\infty^o$. It is easy to see that the right-hand side of (6.8) is independent of the choice of the Iwasawa decomposition $g_\infty = p_\infty k_\infty$.

It is clear that (6.1) is functorial with respect to $\mathcal{Q}$. Let $\tilde{\mathcal{Q}} \supseteq \mathcal{Q}$. Since $\tilde{\mathfrak{q}} \cap \mathfrak{m}_{\mathcal{G}} / \tilde{\mathfrak{q}} \cap \mathfrak{k} = \mathfrak{q} \cap \mathfrak{m}_{\mathcal{G}} / \mathfrak{q} \cap \mathfrak{k}$, the formula

$$\left(i_{\tilde{\mathcal{Q}} \supseteq \mathcal{Q}} \phi \right)(\lambda) = \int_{\boldsymbol{K}_\infty \cap \tilde{\mathcal{Q}}(\mathbb{R})} \phi(k\lambda) \, dk \tag{6.9}$$

for $\lambda \in \Lambda^*(\tilde{\mathfrak{q}} \cap \mathfrak{m}_{\mathcal{G}} / \tilde{\mathfrak{q}} \cap \mathfrak{k})$ and

$$\phi \in \operatorname{Hom}_{\boldsymbol{K}_\infty^o \cap \mathcal{Q}(\mathbb{R})}(\tilde{\mathfrak{q}} \cap \mathfrak{m}_{\mathcal{G}} / \tilde{\mathfrak{q}} \cap \mathfrak{k}, \mathbb{C}) = \operatorname{Hom}_{\boldsymbol{K}_\infty^o \cap \mathcal{Q}(\mathbb{R})}(\mathfrak{q} \cap \mathfrak{m}_{\mathcal{G}} / \mathfrak{q} \cap \mathfrak{k}, \mathbb{C})$$

defines a map

$$i_{\tilde{\mathcal{Q}} \supseteq \mathcal{Q}} \colon \operatorname{Hom}_{\boldsymbol{K}_\infty^o \cap \mathcal{Q}(\mathbb{R})}(\mathfrak{q} \cap \mathfrak{m}_{\mathcal{G}} / \mathfrak{q} \cap \mathfrak{k}, \mathbb{C}) \to \operatorname{Hom}_{\boldsymbol{K}_\infty^o \cap \tilde{\mathcal{Q}}(\mathbb{R})}(\tilde{\mathfrak{q}} \cap \mathfrak{m}_{\mathcal{G}} / \tilde{\mathfrak{q}} \cap \mathfrak{k}, \mathbb{C}).$$

It follows from (6.8) that the isomorphism (6.2) is functorial in $\mathcal{Q}$ if the transition homomorphism for its target is defined by $\tau_{\mathcal{Q}(\mathbb{A}_f)}^{\tilde{\mathcal{Q}}(\mathbb{A}_f)} \otimes i_{\tilde{\mathcal{Q}} \supseteq \mathcal{Q}}$. It is clear that

$$\begin{aligned} i_{\tilde{\mathcal{Q}} \supseteq \mathcal{Q}} & \operatorname{Hom}_{\boldsymbol{K}_\infty^o \cap \mathcal{Q}(\mathbb{R})}(\mathfrak{m}_{\mathcal{Q}} / \mathfrak{q} \cap \mathfrak{k} \otimes \det(\mathfrak{a}_{\mathcal{Q}}^{\mathcal{G}} \oplus \mathfrak{n}_{\mathcal{Q}}), \mathbb{C}) \\ & \subseteq \operatorname{Hom}_{\boldsymbol{K}_\infty^o \cap \tilde{\mathcal{Q}}(\mathbb{R})}(\mathfrak{m}_{\tilde{\mathcal{Q}}} / \tilde{\mathfrak{q}} \cap \mathfrak{k} \otimes \det(\mathfrak{a}_{\tilde{\mathcal{Q}}}^{\mathcal{G}} \oplus \mathfrak{n}_{\tilde{\mathcal{Q}}}), \mathbb{C}). \end{aligned}$$

Therefore, we may define $\boldsymbol{H}(\mathcal{G})_{\tilde{\mathcal{Q}} \supseteq \mathcal{Q}}$ by $\tau_{\mathcal{Q}(\mathbb{A}_f)}^{\tilde{\mathcal{Q}}(\mathbb{A}_f)} \otimes i_{\tilde{\mathcal{Q}} \supseteq \mathcal{Q}}$. To verify that (6.3) is functorial in $\mathcal{Q}$, we have to verify that $p_{\tilde{\mathcal{Q}}} i_{\tilde{\mathcal{Q}} \supseteq \mathcal{Q}}$ vanishes on the kernel of $p_{\mathcal{Q}}$. This follows from the following lemma.

Lemma 6.2. *Let $\mathcal{H}$ be a semisimple algebraic group over $\mathbb{R}$, $\boldsymbol{K} \subset \mathcal{H}(\mathbb{R})$ a maximal compact subgroup, and let $\mathcal{P} = \mathcal{M}\mathcal{A}\mathcal{N}$ be a $\mathbb{R}$-parabolic subgroup of $\mathcal{H}$. Let $\mathfrak{h}$, $\mathfrak{p}$, $\mathfrak{m}$, $\mathfrak{a}$, $\mathfrak{n}$ be the Lie algebras of $\mathcal{H}(\mathbb{R})$, $\mathcal{P}(\mathbb{R})$, $\mathcal{M}(\mathbb{R})$, $\mathcal{A}(\mathbb{R})$, $\mathcal{N}(\mathbb{R})$. If $\lambda \in \Lambda^i \mathfrak{a} \otimes \det \mathfrak{n} \otimes \Lambda^*(\mathfrak{m}/\mathfrak{k} \cap \mathfrak{m}) \subset \Lambda^*(\mathfrak{h}/\mathfrak{h} \cap \mathfrak{k})$ for $i < \dim \mathfrak{a}$, then*

$$\int_{\boldsymbol{K}^o} k\lambda \, dk = 0$$

in $\Lambda^*(\mathfrak{h}/\mathfrak{h} \cap \mathfrak{k})$.

Since (6.1), (6.2) and (6.3) are natural in $\mathcal{Q}$, the same is true for their composition (6.5). Therefore, Proposition 6.1 together with Proposition 5.5 and the proven Borel conjecture imply the following theorem.

Theorem 6.3. *Let $\boldsymbol{H}(\mathcal{G})_{\mathcal{Q}}{}^{*}$ be defined by (6.4), and let $\boldsymbol{H}(\mathcal{G})_{\tilde{\mathcal{Q}}\supseteq\mathcal{Q}} = \tau_{\mathcal{Q}(\mathbb{A}_f)}^{\tilde{\mathcal{Q}}(\mathbb{A}_f)} \otimes i_{\tilde{\mathcal{Q}}\supseteq\mathcal{Q}}$. Then we have an isomorphism of $\mathcal{G}(\mathbb{A}_f)$-modules*

$$H^k(\mathcal{G}, \mathbb{C})_{\mathcal{I}} \cong H^{k-\dim \mathfrak{a}_o^{\mathcal{G}}} \left(C^* \left(\boldsymbol{H}(\mathcal{G})_{\bullet}{}^{*}\right)\right)$$

which respects the canonical real structures on its source and its target.

It remains to prove Lemma 6.2.

Proof (of Lemma 6.2). By Poincare duality, it suffices to verify that

$$\phi\left(\Lambda^j(\mathfrak{a}) \otimes \Lambda^*(\mathfrak{m}/\mathfrak{m}\cap\mathfrak{k})\right) = 0 \tag{6.10}$$

for $j > 0$ and any $\phi \in \mathrm{Hom}_{\boldsymbol{K}^o}(\Lambda^*(\mathfrak{h}/\mathfrak{k}), \mathbb{C})$. Recall the definition of the compact homogeneous space $\boldsymbol{X}_{\mathcal{H}}^{(c)}$ and of the compact duals $\mathcal{H}^{(c)}$, $\mathcal{M}^{(c)}$, and $\mathcal{A}^{(c)}$ from the introduction. Then (6.10) admits a topological reformulation

$$\mathrm{im}\left(H^*(\boldsymbol{X}_{\mathcal{H}}^{(c)}, \mathbb{C}) \to H^*(\boldsymbol{X}_{\mathcal{M}}^{(c)} \times \mathcal{A}^{(c)}(\mathbb{R}), \mathbb{C})\right) \subseteq H^*(\boldsymbol{X}_{\mathcal{M}}^{(c)}, \mathbb{C}) \tag{6.11}$$

in terms of the pull-back of cohomology classes from $\boldsymbol{X}_{\mathcal{H}}^{(c)}$ to $\boldsymbol{X}_{\mathcal{M}}^{(c)} \times \mathcal{A}^{(c)}(\mathcal{R})$. Let J be an integer, and let

$$\begin{aligned} f_J \colon \mathcal{A}^{(c)}(\mathbb{R}) \times \boldsymbol{X}_{\mathcal{M}}^{(c)} &\to \boldsymbol{X}_{\mathcal{H}}^{(c)} \\ f_J(a, x) &= a^J x \end{aligned}$$

be defined by the action of $\mathcal{A}^{(c)}(\mathbb{R})$ on $\boldsymbol{X}_{\mathcal{H}}^{(c)}$ and the embedding $\boldsymbol{X}_{\mathcal{M}}^{(c)} \subset \boldsymbol{X}_{\mathcal{H}}^{(c)}$. To verify (6.11), it suffices to take some $J \neq 0$ and to verify

$$\mathrm{im}(f_J^*) = \mathrm{im}(f_0^*) \tag{6.12}$$

for the pull-back on cohomology with complex coefficients. For the right-hand side of (6.12) is always contained in the right-hand side of (6.11), and for $J \neq 0$ the left-hand sides of (6.11) and (6.12) agree.

As $\mathcal{H}$ was supposed to be semisimple, the fundamental group of $\mathcal{H}^{(c)}(\mathbb{R})$ is finite. Since $\mathcal{A}^{(c)}(\mathbb{R})$ is a product of circles, if J is divisible by a certain positive integer, the map

$$\begin{aligned} \mathcal{A}^{(c)}(\mathbb{R}) &\to \mathcal{H}^{(c)}(\mathbb{R}) \\ a &\to a^J \end{aligned}$$

will be homotopic to the identity. But then f_0 and f_J are homotopic, and this implies (6.12). The proof of Lemma 6.2 is complete. □

For those who are only interested in an algebraic formula for $H^*(\mathcal{G},\mathbb{C})_{\mathcal{I}}$, Theorem 6.3 would be the final result of this paper. It remains to derive the isomorphism (3.4) from this theorem.

Let

$$\check{\boldsymbol{H}}(\mathcal{G})^{\mathcal{Q}*} = C^\infty(\mathcal{Q}(\mathbb{A}_f)\backslash\mathcal{G}(\mathbb{A}_f)) \otimes \mathrm{Hom}_{\boldsymbol{K}_\infty^o\cap\mathcal{Q}(\mathbb{R})}(\Lambda^*(\mathfrak{m}_{\mathcal{Q}}/\mathfrak{m}_{\mathcal{Q}}\cap\mathfrak{k}),\mathbb{C}),$$

where the transition maps $\check{\boldsymbol{H}}(\mathcal{G})^{\mathcal{Q}\supseteq\tilde{\mathcal{Q}}*}$ are given by the embedding

$$C^\infty(\mathcal{Q}(\mathbb{A}_f)\backslash\mathcal{G}(\mathbb{A}_f)) \subseteq C^\infty(\tilde{\mathcal{Q}}(\mathbb{A}_f)\backslash\mathcal{G}(\mathbb{A}_f))$$

and the restriction to $\mathfrak{m}_{\tilde{\mathcal{Q}}}$

$$\mathrm{Hom}_{\boldsymbol{K}_\infty^o\cap\mathcal{Q}(\mathbb{R})}(\Lambda^*(\mathfrak{m}_{\mathcal{Q}}/\mathfrak{m}_{\mathcal{Q}}\cap\mathfrak{k}),\mathbb{C}) \to \mathrm{Hom}_{\boldsymbol{K}_\infty^o\cap\tilde{\mathcal{Q}}(\mathbb{R})}(\Lambda^*(\mathfrak{m}_{\tilde{\mathcal{Q}}}/\boldsymbol{k}\cap\mathfrak{m}_{\tilde{\mathcal{Q}}}),\mathbb{C}).$$

If $\mathcal{P}$ is a standard parabolic subgroup, then $\boldsymbol{K}_\infty^o \cap \mathcal{P}_o(\mathbb{R})$ meets every connected component of $\boldsymbol{K}_\infty \cap \mathcal{P}(\mathbb{R})$ by Proposition 2.1. Consequently, there is a canonical isomorphism between

$$H^*\left(C^*\left(\tilde{\boldsymbol{H}}(\mathcal{G})^{\bullet *}\right)\right)^{\pi_o(\boldsymbol{K}_\infty\cap\mathcal{P}_o(\mathbb{R}))}$$

and the invariants in the hypercohomology of the complex associated to the functor $\boldsymbol{A}(\mathcal{G},\mathbb{C})^{\mathcal{P}}$

$$H^*\left(C^*\left(\tilde{\boldsymbol{H}}(\mathcal{G})^{\bullet *}\right)\right) \cong H_c^*(\mathcal{G},\mathbb{C})_{\mathcal{I}} \cong H^*(\boldsymbol{X}^{(c)}_{\mathcal{M}_{\mathcal{G}}}, C^*(\boldsymbol{A}(\mathcal{G},\mathcal{C}))).$$

This isomorphism identifies the canonical real subspace of its source with

$$i^p H^p(\boldsymbol{X}^{(c)}_{\mathcal{M}_{\mathcal{G}}}, C^*(\boldsymbol{A}(\mathcal{G},\mathcal{C}))).$$

To construct (3.4), we construct a duality between $\tilde{\boldsymbol{H}}(\mathcal{G})^{\bullet *}$ and $\boldsymbol{H}(\mathcal{G})_{\bullet}{}^*$. Let o be an orientation of the real vector space $\mathfrak{m}_{\mathcal{G}}/\mathfrak{k}$. Multiplication by a square root i of -1 defines an isomorphism between $\mathfrak{m}_{\mathcal{G}}/\mathfrak{k}$ and the tangent space of $\boldsymbol{X}^{(c)}_{\mathcal{M}_{\mathcal{G}}}$ at the origin. Therefore, o and i define an orientation o_i of the differentiable manifold $\boldsymbol{X}^{(c)}_{\mathcal{M}_{\mathcal{G}}}$. There exists $\delta_o \in i^{\dim(\mathfrak{m}_{\mathcal{G}}/\mathfrak{k})}\det(\mathfrak{m}_{\mathcal{G}}/\mathfrak{k})$ such that

$$\int^{o_i}_{\boldsymbol{X}^{(c)}_{\mathcal{M}_{\mathcal{G}}}} \delta_o = i^{\dim(\mathfrak{m}_{\mathcal{G}}/\mathfrak{k})}$$

if δ_o is viewed as a real $\dim(\mathfrak{m}_{\mathcal{G}}/\mathfrak{k})$-form on $\boldsymbol{X}^{(c)}_{\mathcal{M}_{\mathcal{G}}}$. We have

$$o_{-i} = (-1)^{\dim(\mathfrak{m}_{\mathcal{G}}/\mathfrak{k})} o_i,$$

hence δ_o is independent of the choice of i. Then δ_o defines a duality

$$\begin{aligned}\mathrm{Hom}_{\boldsymbol{K}_\infty^\circ\cap\mathcal{Q}(\mathbb{R})}(\Lambda^*(\mathfrak{m}_\mathcal{Q}/\mathfrak{m}_\mathcal{Q}\cap\mathfrak{k})\otimes\det(\mathfrak{a}_\mathcal{Q}^\mathcal{G}\oplus\mathfrak{n}_\mathcal{Q}),\mathbb{C})\\ \otimes\mathrm{Hom}_{\boldsymbol{K}_\infty^\circ\cap\mathcal{Q}(\mathbb{R})}(\Lambda^*(\mathfrak{m}_\mathcal{Q}/\mathfrak{m}_\mathcal{Q}\cap\mathfrak{k}),\mathbb{C})\\ \to\mathbb{C}[-\dim(\mathfrak{m}_\mathcal{G}/\mathfrak{k})],\end{aligned}$$

and $\tau_q fQ^{\mathcal{G}(\mathbb{A}_f)}$ defines a duality between $\mathrm{Ind}_{\mathcal{Q}(\mathbb{A}_f)}^{\mathcal{G}(\mathbb{A}_f)}\mathbb{C}$ and $\mathrm{Ind}_{\mathcal{Q}(\mathbb{A}_f)}^{\mathcal{G}(\mathbb{A}_f)}\mathbb{C}_t rQ$. We get a duality

$$\tilde{\boldsymbol{H}}(\mathcal{G})^{\bullet *}\boldsymbol{H}(\mathcal{G})_\bullet{}^*\to\mathbb{C}[-\dim(\mathfrak{m}_\mathcal{G}/\mathfrak{k})] \tag{6.13}$$

which defines an isomorphism (3.4) independent of o; (6.13) changes its sign if o is changed. Furthermore, (6.13) maps the real subspaces of $\boldsymbol{H}$ and $\check{\boldsymbol{H}}$ to $i^{\dim(\mathfrak{m}_\mathcal{G}/\mathfrak{k})}\mathbb{R}$, whence the assertion about real subspaces in Theorem 3.1 applies.

7 Some examples

7.1 Ghost classes in the image of the Borel map

It is rather easy to use the topological model to explicitly compute the kernel of the Borel map

$$I^*_{\mathcal{G}(\mathbb{R}),\boldsymbol{K}_\infty^\circ}\to H^*(\mathcal{G},\mathbb{C}).$$

This allows us to give new examples of ghost classes. Recall that a cohomology class of $\mathcal{G}$ is called a ghost class if it trivially restricts to each boundary component of the Borel–Serre compactification and if its restriction to the full Borel–Serre boundary is not zero. This notion was coined by Borel. The first example of a ghost class was constructed by Harder in the cohomology of GL_3 over totally imaginary fields, using Eisenstein series starting from an algebraic Hecke character whose L-function vanishes at the center of the functional equation. Our computation of the kernel of the Borel map will make it clear that ghost classes abound in the image of the Borel map, at least for most groups of sufficiently high rank.

Recall that $H_c^*(\mathcal{G},\mathbb{C})_\mathcal{I}$ can be computed as the cohomology of the complex of graded vector spaces $C^*(\check{\boldsymbol{H}}^{\bullet *})$. The map

$$C^*(\check{\boldsymbol{H}}^{\bullet *})\to\check{\boldsymbol{H}}^{\mathcal{G}*}\to I^*_{\mathcal{G}(\mathbb{R}),\boldsymbol{K}_\infty^\circ}$$

defines a homomorphism

$$H_o^*(\mathcal{G},\mathbb{C})_\mathcal{I}\to I^*_{\mathcal{G}(\mathbb{R}),\boldsymbol{K}_\infty^\circ} \tag{7.1}$$

which is easily identified with the Poincare dual of the Borel map. It can also be viewed as the restriction to the subspaces which are annihilated by the Hecke ideal $\mathcal{I}$ of the map from cohomology with compact support to L_2-cohomology. By the definition of the differential of the complex $C^*(\check{\boldsymbol{H}}^{\bullet *})$, the image of (7.1) is the space

$$(I^*_{\mathcal{G}(\mathbb{R}),\boldsymbol{K}^o_\infty})_{\text{Image}} = \ker\left(I^*_{\mathcal{G}(\mathbb{R}),\boldsymbol{K}^o_\infty} \to \bigoplus_{\substack{\mathcal{P}\in\mathfrak{P} \\ \dim \mathfrak{a}^{\mathcal{G}}_{\mathcal{P}}=1}} I^*_{\mathcal{M}_{\mathcal{P}}(\mathbb{R}),\boldsymbol{K}^o_\infty\cap\mathcal{P}(\mathbb{R})} \right). \tag{7.2}$$

In other words, a cohomology class of the constant representation of $\mathcal{G}$ is in the image of the cohomology with compact support if and only if its restriction to the cohomology of the constant representation of any maximal Levi component vanishes. By Poincare duality, the kernel $(I^*_{\mathcal{G}(\mathbb{R}),\boldsymbol{K}^o_\infty})_{\text{Kernel}}$ of the Borel map is equal to the orthogonal complement of $(I^*_{\mathcal{G}(\mathbb{R}),\boldsymbol{K}^o_\infty})_{\text{Image}}$. Let $(I^*_{\mathcal{G}(\mathbb{R}),\boldsymbol{K}^o_\infty})_{\text{Ghost}}$ be the space of all invariant forms $i \in I^*_{\mathcal{G}(\mathbb{R}),\boldsymbol{K}^o_\infty}$ such that, for any parabolic subgroup $\mathcal{P}$ with $\dim \mathfrak{a}^{\mathcal{G}}_{\mathcal{P}} = 1$, the image of j in

$$I^*_{\mathcal{M}_{\mathcal{P}}(\mathbb{R}),\boldsymbol{K}^o_\infty\cap\mathcal{P}(\mathbb{R})}$$

belongs to

$$(I^*_{\mathcal{M}_{\mathcal{P}}(\mathbb{R}),\boldsymbol{K}^o_\infty\cap\mathcal{P}(\mathbb{R})})_{\text{Kernel}}.$$

Then the space of ghost classes in the image of the Borel map is isomorphic to $(I_{\text{Ghost}}/I_{\text{Image}} + I_{\text{Kernel}})^*_{\mathcal{G}(\mathbb{R}),\boldsymbol{K}^o_\infty}$. This follows from: the fact ([20, 1.10]) that after identifying $(\mathfrak{g},\boldsymbol{K})$- and de Rham-cohomology, the homomorphism defined on $(\mathfrak{g},\boldsymbol{K})$-cohomology, by taking the constant term along $\mathcal{P}$, corresponds to restriction to the Borel–Serre boundary component belonging to $\mathcal{P}$; and (7.3) as presented below in Proposition 7.2.

Let us explain this a little more for the case of groups over totally imaginary fields. That is, let $\mathcal{G}$ be obtained by Weil restriction from a totally imaginary field. Then $\boldsymbol{X}^{(c)}_{\mathcal{G}}$ has a group structure. Therefore, its cohomology $I^*_{\mathcal{G}(\mathbb{R}),\boldsymbol{K}^o_\infty}$ is a Hopf algebra. By the Hopf structure theorem, it is an exterior algebra over a graded space $E^*(\mathcal{G})$ of primitive elements, which are of odd order. The same is true for all Levi components of parabolic subgroups of $\mathcal{G}$. Let

$$E^*_{\text{Top}}(\mathcal{G}) = \ker\left(E^*(\mathcal{G}) \to \bigoplus_{\substack{\mathcal{P}\in\mathfrak{P} \\ \dim \mathfrak{a}^{\mathcal{G}}_{\mathcal{P}}=1}} E^*(\mathcal{M}_{\mathcal{G}}) \right)$$

and

$$E^*_{\text{Ghost}}(\mathcal{G}) = \ker\left(E^*(\mathcal{G}) \to \bigoplus_{\substack{\mathcal{P}\in\mathfrak{P} \\ \dim \mathfrak{a}^{\mathcal{G}}_{\mathcal{P}}=2}} E^*(\mathcal{M}_{\mathcal{G}}) \right).$$

Then

$$\begin{aligned}
\big(I^*_{\mathcal{G}(\mathbb{R}),\boldsymbol{K}^o_\infty}\big)_{\text{Image}} &= E^*_{\text{Top}}(\mathcal{G}) \wedge \Lambda^*\big(E^*(\mathcal{G})\big) \\
\big(I^*_{\mathcal{G}(\mathbb{R}),\boldsymbol{K}^o_\infty}\big)_{\text{Kernel}} &= \det\big(E^*_{\text{Top}}(\mathcal{G})\big) \wedge \Lambda^*\big(E^*(\mathcal{G})\big) \\
\big(I^*_{\mathcal{G}(\mathbb{R}),\boldsymbol{K}^o_\infty}\big)_{\text{Ghost}} &= \det\big(E^*_{\text{Ghost}}(\mathcal{G})/E^*_{\text{Top}}(\mathcal{G})\big) \wedge \Lambda^*\big(E^*(\mathcal{G})/E^*_{\text{Top}}(\mathcal{G})\big).
\end{aligned}$$

For instance, for SL_n over a totally imaginary field $\mathbb{K}$, we have primitive generators $\lambda_2^{(v)},\dots,\lambda_n^{(v)}$ for each v in the set $\mathfrak{V}_{\mathbb{K}}^{\infty}$ of archimedean primes of $\mathbb{K}$, with the relation $\sum_{\mathfrak{V}_{\mathbb{K}}^{\infty}} \lambda_1^{(v)} = 0$. The degree of $\lambda_j^{(v)}$ is $2j-1$. The following fact is an obvious consequence of this discussion.

Theorem 7.1. *Then an invariant form is in the image of cohomology with compact support if and only if it is a sum of monomials which contain one of the classes $\lambda_n^{(v)}$. It is in the kernel of the Borel map if and only if it is divisible by $\bigwedge_{\mathfrak{V}_{\mathbb{K}}^{\infty}} \lambda_n^{(v)}$. It defines a ghost class if and only if it is a sum of monomials which contain all of the classes $\lambda_{n-1}^{(v)}$ but none of the classes $\lambda_n^{(v)}$.*

The space $E^*(\mathcal{G})$ is known for groups over totally imaginary fields by the known calculation of the cohomology of compact Lie groups. (See [1, §11] for a statement of the result and for references, and [12, §VI.7] for the case of the classical groups.) Therefore, the spaces $(I^*_{\mathcal{G}(\mathbb{R}),\boldsymbol{K}_\infty^\circ})_{\text{Image, Kernel, Ghost}}$ are at least in principle known for groups over totally imaginary fields.

Let us also formulate the result about the kernel of the Borel map and about ghost classes for SL_n over a field $\mathbb{K}$ which has real places. We first have to formulate the necessary facts about the cohomology of $\mathrm{SU}(n,\mathbb{R})/\mathrm{SO}(n,\mathbb{R})$. They can be obtained from the consideration of the Leray spectral sequence for the projection $\mathrm{SU}(n,\mathbb{R}) \to \mathrm{SU}(n,\mathbb{R})/\mathrm{SO}(n,\mathbb{R})$, either by hand or by the general theory (cf. [12, XI.4.4.]).

Proposition 7.2. *If n is odd, then the cohomology with complex coefficients of $\mathrm{SU}(n,\mathbb{R})/\mathrm{SO}(n,\mathbb{R})$ is an exterior algebra with generators $\tilde\lambda_3, \tilde\lambda_5,\dots,\tilde\lambda_n$, where $\deg\tilde\lambda_i = 2i-1$. Furthermore, $\tilde\lambda_i$ can be obtained from the primitive element λ_i in the cohomology of $\mathrm{SU}(n,\mathbb{R})$ by pull-back via the map*

$$\begin{aligned} \mathrm{SU}(n,\mathbb{R})/\mathrm{SO}(n,\mathbb{R}) &\to \mathrm{SU}(n,\mathbb{R}) \\ \dot g &\to g\cdot g^{\mathrm{T}}. \end{aligned} \tag{7.3}$$

If n is even, then the cohomology of $\mathrm{SU}(n,\mathbb{R})/\mathrm{SO}(n,\mathbb{R})$ is an exterior algebra generated by elements $\tilde\lambda_3,\dots,\tilde\lambda_{n-1}$ obtained in the same way as above, and by a class ε in degree n, which is the Euler class of the canonical n-dimensional orientable real bundle on $\mathrm{SU}(n,\mathbb{R})/\mathrm{SO}(n,\mathbb{R})$.

If $\sum_{i=1}^k n_i \le n$, then the restriction of $\tilde\lambda_l$ to

$$\prod_{i=1}^k \mathrm{SU}(n_i,\mathbb{R})/\mathrm{SO}(n_i,\mathbb{R}) \subset \mathrm{SU}(n,\mathbb{R})/\mathrm{SO}(n,\mathbb{R}) \tag{7.4}$$

is

$$\sum_{\substack{1\le i\le k\\ n_i\le l}} \tilde\lambda_l^{(i)},$$

where $\tilde\lambda_l^{(i)}$ is the copy of $\tilde\lambda_l$ for the i-th factor in (7.4). If n is even, then the restriction of the Euler class ε to (7.4) can be described as follows.

If $n = \sum_{i=1}^{k} n_i$ *and if all the* n_i *are even, then the restriction of* ε *is given by*

$$\varepsilon^{(1)} \wedge \dots \varepsilon^{(k)},$$

where ε^i *is the copy of* ε *for the i-th factor in (7.4). If* $n < \sum_{i=1}^{k} n_i$ *or if some of the* n_i *are odd, then the restriction of the Euler class is zero.*

Now let $\mathbb{K}$ be a field which has at least one real place. Let $\mathcal{G}$ be SL_n over $\mathbb{K}$. If n is odd, then the space of invariant forms is an exterior algebra with generators $\tilde{\lambda}_3^{(u)}$, $\tilde{\lambda}_5^{(u)}$,..., $\tilde{\lambda}_n^{(u)}$ for the real places u and $\lambda_2^{(v)}$,...,$\lambda_n^{(v)}$ for the complex places v (if there are any complex places). A monomial in these generators belongs to $(I^*_{\mathcal{G}(\mathbb{R}),\boldsymbol{K}^\circ_\infty})_{\mathrm{Image}}$ if and only if it contains one of the generators $\tilde{\lambda}_n^{(u)}$ for a real place u or one of the generators $\lambda_n^{(v)}$ for a complex place v. It belongs to the kernel of the Borel map if and only if it is divisible by

$$\bigwedge_{u \text{ real}} \tilde{\lambda}_n^{(u)} \wedge \bigwedge_{v \text{ imaginary}} \lambda_n^{(v)}.$$

If n is even, then $I^*_{\mathcal{G}(\mathbb{R}),\boldsymbol{K}^\circ_\infty}$ is an exterior algebra with generators $\tilde{\lambda}_3^{(u)}$, $\tilde{\lambda}_5^{(u)}$,..., $\tilde{\lambda}_{n-1}^{(u)}$ and $\varepsilon^{(u)}$ for each real place u and $\lambda_2^{(v)}$,..., $\lambda_n^{(v)}$ for the complex places v. A monomial in these generators belongs to $(I^*_{\mathcal{G}(\mathbb{R}),\boldsymbol{K}^\circ_\infty})_{\mathrm{Image}}$ if and only if it contains one of the following factors:

- $\lambda_n^{(v)}$ for a imaginary place v;
- $\varepsilon^{(u)} \wedge \tilde{\lambda}_{n-1}^{(w)}$ for real places u and w;
- or $\varepsilon^{(u)} \wedge \lambda_{n-1}^{(v)}$ for a real place u and an imaginary place v.

A monomial belongs to the kernel of the Borel map if and only if it contains at least one of the following two factors:

$$\bigwedge_{u \text{ real}} \tilde{\lambda}_{n-1}^{(u)} \wedge \bigwedge_{v \text{ imaginary}} (\tilde{\lambda}_n^{(u)} \wedge \tilde{\lambda}_{n-1}^{(u)})$$

$$\text{or } \bigwedge_{u \text{ real}} \varepsilon^{(u)} \wedge \bigwedge_{v \text{ imaginary}} \lambda_n^{(v)},$$

where a product over the set of imaginary places is supposed to be one if the field is totally real. In particular, if $n > 2$ is even and if $\mathbb{K}$ is totally real, then $(I^*_{\mathcal{G}(\mathbb{R}),\boldsymbol{K}^\circ_\infty})_{\mathrm{Image}}$ does not contain $(I^*_{\mathcal{G}(\mathbb{R}),\boldsymbol{K}^\circ_\infty})_{\mathrm{Kernel}}$ completely.

We can use this to describe all ghost classes in the image of the Borel map. If n is odd, then a monomial in the generators of $I^*_{\mathcal{G}(\mathbb{R}),\boldsymbol{K}^\circ_\infty}$ is a ghost class if and only if it contains all the generators $\tilde{\lambda}_{n-2}^{(u)}$ for all the real places u and all the generators $\lambda_{n-1}^{(v)}$ and $\lambda_{n-2}^{(v)}$ for all the imaginary places v, but none of the generators $\tilde{\lambda}_n^{(u)}$ or $\lambda_n^{(v)}$. If $n = 3$, this means that there are no ghost classes in the image of the Borel map. (Recall our assumption that $\mathbb{K}$ is not purely imaginary.)

If n is even, then a monomial μ in the generators of $I^*_{\mathcal{G}(\mathbb{R}),\boldsymbol{K}^o_\infty}$ defines a ghost class if and only if at least one of the following four conditions is satisfied:

- $\mathbb{K}$ is not totally real, and μ contains all the generators $\lambda^{(v)}_{n-1}$ for v complex and $\tilde{\lambda}^{(u)}_{n-1}$ for u imaginary, but none of the classes $\lambda^{(v)}_n$ nor any Euler class $\varepsilon^{(u)}$;
- $\mathbb{K}$ is not totally real, and μ contains all the generators $\varepsilon^{(u)}$ for u real and $\lambda^{(v)}_{n-2}$ for v imaginary, but none of the generators $\lambda^{(v)}_n$;
- $n \geq 6$ and $\mathbb{K}$ is not totally real, and μ contains at least one of the generators $\varepsilon^{(u)}$ and all of the generators $\tilde{\lambda}^{(u)}_{n-3}$, $\lambda^{(v)}_{n-3}$ and $\lambda^{(v)}_{n-2}$, but none of $\lambda^{(v)}_n$, $\lambda^{(v)}_{n-1}$ or $\tilde{\lambda}^{(u)}_{n-1}$;
- or $n \geq 6$ and $\mathbb{K} \neq \mathbb{Q}$ is totally real, and μ contains at least one but not all of the generators $\varepsilon^{(u)}$ and all of the generators $\tilde{\lambda}^{(u)}_{n-3}$, but none of $\tilde{\lambda}^{(u)}_{n-1}$.

If $n = 4$ and $\mathbb{K}$ is totally real or if $n > 4$ is even and $\mathbb{K} = \mathbb{Q}$, this means that there are no ghost classes in the image of the Borel map.

In our description of ghost classes, we have used the following fact[1].

Proposition 7.3. *Let $\mathcal{P}$ be a standard parabolic subgroup. Then the image of the restriction map*

$$H^*_{\mathfrak{m}_\mathcal{G},\boldsymbol{K}_\infty}(\mathbb{C}) \to H^*_{\mathfrak{m}_\mathcal{P}+\mathfrak{n},\boldsymbol{K}_\infty\cap\mathcal{P}(\mathbb{R})}(\mathbb{C})$$

is contained in $H^*_{\mathfrak{m}_\mathcal{P}+\mathfrak{n}_\mathcal{P},\boldsymbol{K}_\infty\cap\mathcal{P}(\mathbb{R})}(\mathbb{C}) \subset H^*_{\mathfrak{m}_\mathcal{P}+\mathfrak{n},\boldsymbol{K}_\infty\cap\mathcal{P}(\mathbb{R})}(\mathbb{C})$.

Proof. By an easy induction argument, it suffices to prove this assertion for maximal proper $\mathbb{Q}$-parabolic subgroups. In this case, it follows from Lemma 7.4 below that

$$H^*_{\mathfrak{m}_\mathcal{P}+\mathfrak{n},\boldsymbol{K}_\infty\cap\mathcal{P}(\mathbb{R})}(\mathbb{C}) \cong H^*_{\mathfrak{m}_\mathcal{P}+\mathfrak{n},\boldsymbol{K}_\infty\cap\mathcal{P}(\mathbb{R})}(\mathbb{C}) \oplus H^{*-\dim\mathfrak{n}_\mathcal{P}}_{\mathfrak{m}_\mathcal{P}+\mathfrak{n},\boldsymbol{K}_\infty\cap\mathcal{P}(\mathbb{R})}(\det\mathfrak{n}_\mathfrak{P}),$$

and it follows from (6.2) that the restriction of an element of $H^*_{\mathfrak{m}_\mathcal{G},\boldsymbol{K}_\infty}(\mathbb{C})$ never has a nonvanishing projection to the second summand. □

Lemma 7.4. *Let $\mathcal{P}$ be a maximal proper $\mathbb{Q}$-parabolic subgroup of $\mathcal{G}$. Then*

$$H^*(\mathfrak{n}_\mathcal{P},\mathbb{C})^{\mathcal{M}_\mathcal{P}} = \mathbb{C} \oplus \det\mathfrak{n}_\mathcal{P}[-\dim\mathfrak{n}_\mathcal{P}]. \tag{7.5}$$

Proof. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ which contains $\mathfrak{a}_o$ and is contained in $\mathfrak{a}_o \oplus \mathfrak{m}_\mathcal{P}$. Let $\mathcal{B} \subseteq \mathcal{P}_o$ be a Borel subgroup defined over $\mathbb{C}$ with $\mathfrak{h} \subseteq \mathfrak{b}$, and let Δ_η be the set of simple positive roots of $\mathfrak{h}$ determined by B. This set decomposes according to the restrictions to $\mathfrak{a}_o$:

$$\Delta_\mathfrak{h} = \bigcup_{\mathfrak{a}\in\Delta_o\cap\{0\}} \Delta_{\mathfrak{h},\alpha}.$$

[1] I am indebted to A. Kewenig and T. Rieband for pointing out that this is not self-evident.

By a theorem of Kostant ([27, Theorem 9.6.2] or [26, Theorem 3.2.3])

$$H^*(\mathfrak{n}_{\mathcal{P}}, \mathbb{C}) \cong \sum_{\substack{w \in \Omega(\mathfrak{h},\mathfrak{g}) \\ w^{-1}\Delta_{\mathfrak{h}}^{\mathfrak{m}_{\mathcal{P}}} > 0}} F_{w\rho_h - \rho_h}[-\ell(w)]. \tag{7.6}$$

In fact, it follows from the proof given in the above references that (7.6) even holds in the derived category of $(\mathfrak{m}_{\mathcal{P}}, \boldsymbol{K}_\infty \cap \mathcal{P}(\mathbb{R}))$-modules. This implies the splitting of the Leray spectral sequence, which will be used below. Since both summands on the right-hand side of (7.5) are accounted for by this formula, it suffices to show that there are at most two w for which the corresponding summand in (7.6) contributes to (7.5).

Indeed, if the summand belonging to w in (7.6) contributes to (7.5), then

$$\langle \check{\alpha}, w\rho_{\mathfrak{h}} \rangle = \langle \check{\alpha}, \rho_{\mathfrak{h}} \rangle = 1 \tag{7.7}$$

for all $\alpha \in \Delta_{\mathfrak{h}}^{\mathfrak{m}_{\mathcal{P}}}$ and

$$\langle \check{\alpha}, w\rho_{\mathfrak{h}} \rangle = \langle \check{\beta}, w\rho_{\mathfrak{h}} \rangle \tag{7.8}$$

for $\alpha, \beta \in \Delta_{\mathfrak{h},\gamma}$, where $\Delta_o = \Delta_o^{\mathcal{P}} \cup \{\gamma\}$. The first of these conditions implies that $w^{-1}\alpha$ is not only positive but also a simple positive root. It follows from the second condition, (7.8), either $w^{-1}\Delta_{\mathfrak{h},\gamma} > 0$ or $w^{-1}\Delta_{\mathfrak{h},\gamma} < 0$. In the first case, w^{-1} maps every positive root to a positive root, and w is the identity. In the second case, let γ be a root of $\mathfrak{h}$ in $\mathfrak{n}_{\mathcal{P}}$. Then $\gamma = \gamma' + \gamma''$, where γ' is a linear combination of elements of $\Delta_{\mathfrak{h},\gamma}$ with nonnegative coefficients, and γ'' is a linear combination of elements of $\Delta_{\mathfrak{h}}^{\mathcal{P}}$. By our assumption, $w^{-1}\gamma'$ is a linear combination of simple roots with nonpositive coefficients. Since γ' does not vanish on $\mathfrak{a}_{\mathcal{P}}$, it is not a linear combination of elements of $\Delta_{\mathfrak{h}}^{\mathcal{P}}$. Therefore, there is an element $\alpha \in \Delta_{\mathfrak{h}} - w^{-1}\Delta_{\mathfrak{h}}^{\mathcal{P}}$ which occurs with a negative coefficient in the representation of $w^{-1}\gamma'$ as a linear combination of the elements of $\Delta_{\mathfrak{h}}$. Since $w^{-1}\gamma''$ is a linear combination of the elements of $w^{-1}\Delta_{\mathfrak{h}}^{\mathcal{P}} \subset \Delta_{\mathfrak{h}}$, this means that α occurs with a negative coefficient in the representation of $w^{-1}\gamma$ as a linear combination of positive roots. But this means that w^{-1} maps all positive roots of $\mathfrak{h}$ which do not occur in $\mathfrak{l}_{\mathcal{P}}$ to negative roots. Therefore, the length of w is the largest possible, and the contribution of w to (7.6) is in the highest possible degree, which is one-dimensional and coincides with the second summand in (7.5) □

7.2 SL_n over imaginary quadratic fields

Let $\mathbb{K}$ be an imaginary quadratic field, and let $\mathcal{G} = \mathrm{res}_{\mathbb{Q}}^{\mathbb{K}} \mathrm{SL}_n$. We want to explicitly compute $H_c^*(\mathcal{G}, \mathbb{C})_{\mathcal{I}}$. We will directly use the complex $C^*\left(\check{\boldsymbol{H}}(\mathcal{G})^{\bullet *}\right)$. Let us first describe this complex explicitly.

Recall that the cohomology of the constant representation of SL_n over an imaginary quadratic field is the exterior algebra with generators $\lambda_2, \ldots, \lambda_n$.

The degree of λ_n is $2n-1$, and, using the coalgebra structure of the cohomology of $\boldsymbol{X}_{\mathcal{G}}^{(c)} = \mathrm{SU}(n,\mathbb{C})$ coming from the group law, λ_n is characterised up to multiplication by a nonvanishing number as the primitive element in degree $2n-1$. We will assume that for $k \leq l < n$, the restriction of λ_k on SL_n is λ_k on SL_l.

Let the minimal parabolic subgroup be the stabilizer of the full flag $V_1 \subset V_2 \subset \ldots \subset V_n = \mathbb{K}^n$. Then any standard parabolic subgroup $\mathcal{P}$ is the stabilizer of a flag $V_{i_1} \subset V_{i_2} \subset \cdots \subset V_{i_K}$ for some sequence $0 < i_1 < i_2 < \cdots < i_K = n$. Then

$$\mathcal{M}_{\mathcal{P}} = \prod_{l=1}^{K} \mathrm{Res}_{\mathbb{Q}}^{\mathbb{K}} \mathrm{SL}_{i_l - i_{l-1}}, \qquad i_0 \colon = 0$$

hence the cohomology of $\boldsymbol{X}_{\mathcal{M}_{\mathcal{P}}}^{(c)}$ is an exterior algebra with generators

$$\lambda_2^{(1)}, \ldots, \lambda_{i_1}^{(1)}, \lambda_2^{(2)}, \ldots, \lambda_{i_2-i_1}^{(2)}, \ldots, \lambda_2^{(K)}, \ldots, \lambda_{i_K - i_{K-1}}^{(K)},$$

where the supercript $^{(l)}$ stands for the l-th simple factor of the Levi component. If $i_l - i_{l-1} = 1$, there is primitive element of $H^*(\mathcal{M}_{\mathcal{G}})$ belonging to the l-th factor. Furthermore, the restriction from $\boldsymbol{X}_{\mathcal{G}}^{(c)}$ to $\boldsymbol{X}_{\mathcal{M}_{\mathcal{P}}}^{(c)}$ of the primitve generator λ_k is given by

$$\mathrm{res}_{\boldsymbol{X}_{\mathcal{M}_{\mathcal{P}}}^{(c)}}^{\boldsymbol{X}_{\mathcal{G}}^{(c)}} \lambda_k = \sum_{i_l - i_{l-1} \geq k} \lambda_k^{(l)}. \tag{7.9}$$

Finally,

$$\boldsymbol{K}_\infty \cap \mathcal{P}(\mathbb{R}) = \boldsymbol{S}(\boldsymbol{U}(i_1) \times \boldsymbol{U}(i_2 - i_1) \times \cdots \times \boldsymbol{U}(i_K - i_{K-1}))$$

is connected, hence its group of connected components does not interfere with the computation of the functor $\check{\boldsymbol{H}}(\mathcal{G})^{\bullet *}$. Therefore, we get an explicit description of the functor $\check{\boldsymbol{H}}(\mathcal{G})^{\bullet *}$ which we now want to describe.

Let $\mathfrak{L}_n(\mathbb{K})$ be the set of functions

$$\mathfrak{l} \colon \{2, \ldots, n\} \to \{0, 1, \ldots\}$$

such that

$$\sum_{j=1}^{\infty} \max \{ k \,|\, \mathfrak{l}(k) \geq j \} \leq n. \tag{7.10}$$

If the parabolic subgroup $\mathcal{P}$ corresponds to $0 < i_1 < \cdots < i_K = n$, let $\mathfrak{Y}_{\mathfrak{l},\mathcal{P}}$ be the set of functions

$$\mathfrak{y} \colon \{(k,l) \,|\, 2 \leq k \leq n, 1 \leq l \leq \mathfrak{l}(k) \} \to \{1, \ldots, k\}$$

with the property that

$$i_{\mathfrak{y}(k,l)} - i_{\mathfrak{y}(k,l)-1} \geq k \tag{7.11}$$

and

$$0 < \mathfrak{y}(k,1) < \mathfrak{y}(k,2) < \cdots < \mathfrak{y}(k,\mathfrak{l}(k)). \tag{7.12}$$

Note that $\mathfrak{Y}_{\mathfrak{l},\mathcal{P}}$ is not functorial with respect to $\mathcal{P}$. Let $\tilde{\mathcal{P}} \supseteq \mathcal{P}$, $\mathfrak{y} \in \mathfrak{Y}_{\mathfrak{l},\mathcal{P}}$, $\tilde{\mathfrak{y}} \in \mathfrak{Y}_{\mathfrak{l},\tilde{\mathcal{P}}}$. Let $\mathcal{P}$ belong to the sequence $0 < i_1 < \cdots < i_K = n$ and let $\tilde{\mathcal{P}}$ correspond to $0 < \tilde{i}_1 < \cdots < \tilde{i}_{\tilde{K}} = n$. Then $\{\tilde{i}_1, \ldots, \tilde{i}_{\tilde{K}}\} \subseteq \{i_1, \ldots, i_K\}$. We will write $\tilde{\mathfrak{y}} \trianglerighteq \mathfrak{y}$ if

$$\tilde{i}_{\tilde{\mathfrak{y}}(k,l)-1} < i_{\mathfrak{y}(k,l)} \leq \tilde{i}_{\mathfrak{y}(k,l)}. \tag{7.13}$$

It is clear that for given $\mathcal{P}$, $\tilde{\mathcal{P}}$, and $\mathfrak{y}$ there is at most one $\tilde{\mathfrak{y}}$ with $\tilde{\mathfrak{y}} \trianglerighteq \mathfrak{y}$. Let $\boldsymbol{I}_{\mathfrak{l}}^{\mathcal{P}}$ be the vector space with base $\mathfrak{Y}_{\mathfrak{l},\mathcal{P}}$. Then $\boldsymbol{I}_{\mathfrak{l}}^{\mathcal{P}}$ is a contravariant functor from $\mathfrak{P}$ to the category of vector spaces if we put for $\tilde{\mathfrak{y}} \in \mathfrak{Y}_{\mathfrak{l},\tilde{\mathcal{P}}} \subset (\boldsymbol{I}_{\mathcal{P}})_{\mathfrak{l}}$

$$\boldsymbol{I}_{\mathfrak{l}}^{\tilde{\mathcal{P}} \supseteq \mathcal{P}}(\tilde{\mathfrak{y}}) = \sum_{\substack{\mathfrak{y} \in \mathfrak{Y}_{\mathfrak{l},\mathcal{P}} \\ \tilde{\mathfrak{y}} \trianglerighteq \mathfrak{y}}} \mathfrak{y}. \tag{7.14}$$

By (7.9), the map

$$\boldsymbol{I}_{\mathfrak{l}}^{\mathcal{P}}[-\deg \mathfrak{l}] \to \check{\boldsymbol{H}}(\mathcal{G})^{\mathcal{P}*}$$
$$\mathfrak{y} \to \bigwedge_{k=2}^{n} \bigwedge_{l=1}^{\mathfrak{l}(k)} \lambda_k^{\mathfrak{y}(k,l)},$$

where

$$\deg \mathfrak{l} = \sum_{k=2}^{n} (2k-1)\mathfrak{l}(k) \tag{7.15}$$

is a functormorphism. We get a direct sum decomposition

$$\check{\boldsymbol{H}}(\mathcal{G})^{\mathcal{P}*} \cong \bigoplus_{\mathfrak{l} \in \mathfrak{L}_n(\mathbb{K})} \boldsymbol{I}_{\mathfrak{l}}^{\mathcal{P}}[-\deg \mathfrak{l}] \otimes C^{\infty}(\mathcal{P}(\mathbb{A}_f)\backslash\mathcal{G}(\mathbb{A}_f)). \tag{7.16}$$

If $\boldsymbol{K}_f$ is a good maximal compact subgroup of $\mathcal{G}(\mathbb{A}_f)$, we also get a direct sum decomposition for spherical vectors

$$(\check{\boldsymbol{H}}(\mathcal{G})^{\mathcal{P}*})^{\boldsymbol{K}_f} \cong \bigoplus_{\mathfrak{l} \in \mathfrak{L}_n(\mathbb{K})} \boldsymbol{I}_{\mathfrak{l}}^{\mathcal{P}}[-\deg \mathfrak{l}]. \tag{7.17}$$

Let us first formulate our result for spherical vectors in the cohomology.

Theorem 7.5. *For $\mathfrak{l} \in \mathfrak{L}_n(\mathbb{K})$, $\mathfrak{e} \in \{0,1\}$, and $N \leq 0$, let $\mathfrak{X}_{N,\mathfrak{e},\mathfrak{l}}$ be the set of ordered $(N+1)$-tuples $\mathfrak{x} = (X_0, \ldots, X_N)$ of subsets of $\{2, \ldots, n\}$ with the following properties:*

- *Each number k with $2 \leq k \leq n$ belongs to precisely $\mathfrak{l}(k)$ of the sets X_i;*
- *We have*

$$\sum_{i=0}^{N} \max \#\{X_i\} = n - \mathfrak{e}.$$

If $\mathfrak{l} = 0$, we put $\mathfrak{X}_{N,\mathfrak{e},\mathfrak{l}} = \emptyset$. Then for each $\mathfrak{x} \in \mathfrak{X}_{N,\mathfrak{e},\mathfrak{l}}$, $H^*(C^*(\boldsymbol{I}_{\mathfrak{l}}^\bullet))$ has a generator $\{\mathfrak{x}\}$ in degree $N + \mathfrak{e}$, and we have

$$H^i(C^*(\boldsymbol{I}_{\mathfrak{l}}^\bullet)) = \bigoplus_{\mathfrak{e}=0}^{1} \bigoplus_{\mathfrak{x}\in\mathfrak{X}_{i-\mathfrak{e},\mathfrak{e},\mathfrak{l}}} \mathbb{C}\cdot\{\mathfrak{x}\}. \tag{7.18}$$

Consequently,

$$H_c^j(\mathcal{G},\mathbb{C})_{\mathcal{I}} \cong \bigoplus_{\mathfrak{l}\in\mathfrak{L}_n(\mathbb{K})} \bigoplus_{\mathfrak{e}=0}^{1} \bigoplus_{\mathfrak{x}\in\mathfrak{X}_{j-\mathfrak{e}-\deg\mathfrak{l}}} \mathbb{C}\cdot\{\mathfrak{x}\}.$$

Moreover, let the ordering $\prec$ on the roots which was used to define the complex $C^*(\boldsymbol{F}^\bullet)$ be

$$x_1 - x_2 \prec x_2 - x_3 \prec \cdots \prec x_{n-1} - x_n.$$

Then for $\mathfrak{x} = (X_0,\ldots,X_N) \in \mathfrak{X}_{N,0,\mathfrak{l}}$ a representative of the cohomology class $\{\mathfrak{x}\}$ is given by the element

$$\bigwedge_{i=0}^{N} \bigwedge_{\substack{j=2\\ j\in X_i}}^{\#(X_i)} \lambda_j^{(i)} \tag{7.19}$$

in the cohomology of $\boldsymbol{X}^{(c)}_{\mathcal{M}_{\mathcal{P}}}$, where $\mathcal{P} \in \mathfrak{P}$ is the stabilizer of the standard flag of vector spaces with dimensions

$$0 < \#(X_0) < \#(X_0) + \#(X_1) < \cdots < \sum_{i=0}^{N-2} \#(X_i) < \sum_{i=0}^{N-1} \#(X_i) = n.$$

If $\mathfrak{x} = (X_0,\ldots,X_N) \in \mathfrak{X}_{N,1,\mathfrak{l}}$ and if $0 \le k \le N+1$, then a representative of the cohomology class $\{\mathfrak{x}\}$ is given by the element

$$(-1)^k \bigwedge_{i=0}^{k-1} \bigwedge_{\substack{j=2\\ j\in X_i}}^{\#(X_i)} \lambda_j^{(i)} \wedge \bigwedge_{i=k}^{N} \bigwedge_{\substack{j=2\\ j\in X_i}}^{\#(X_i)} \lambda_j^{(i+1)} \tag{7.20}$$

in the cohomology of $\boldsymbol{X}^{(c)}_{\mathcal{M}_{\mathcal{P}}}$, where $\mathcal{P} \in \mathfrak{P}$ is the stabilizer of the standard flag of vector spaces with dimensions

$$0 < \#(X_0) < \#(X_0) + \#(X_1) < \cdots < \sum_{i=0}^{k-1} \#(X_i)$$
$$< 1 + \sum_{i=0}^{k-1} \#(X_i) < \cdots 1 + \sum_{i=1}^{N-2} \#(X_i) < 1 + \sum_{i=1}^{n-1} \#(X_i) = n.$$

For instance, if $n = 2$, then the only spherical vector in $H_c^*(\mathcal{G}, \mathbb{C})_{\mathcal{I}}$ is the volume form in degree 2. This is in good keeping with the results of Harder for SL_2 and also with the computation of R. Staffeldt [25, Theorem IV.1.3.] which implies that $H^*(\mathrm{SL}_2(\mathbb{Z}[i]), \mathbb{C})$ vanishes in positive dimension. In particular, there are no harmonic cusp forms for $\mathrm{SL}_2(\mathbb{Z}[i])$. If $n = 3$, then $H_c^*(\mathcal{G}, \mathbb{C})_{\mathcal{I}}$ contains the following three spherical vectors:

- In degree 4, the cohomology class belonging to
$$\mathfrak{l}(k) = \begin{cases} 1 \text{ if } k = 2 \\ 0 \text{ otherwise} \end{cases}$$
and $\mathfrak{x} = \{\{2\}\} \in \mathfrak{X}_{0,1,\mathfrak{l}}$;
- In degree 5, the cohomology class belonging to
$$\mathfrak{l}(k) = \begin{cases} 1 \text{ if } k = 3 \\ 0 \text{ otherwise} \end{cases}$$
and $\mathfrak{x} = \{\{3\}\} \in \mathfrak{X}_{0,0,\mathfrak{l}}$ (This class maps to λ_3 in $I^*_{\mathcal{G}(\mathbb{R}),\boldsymbol{K}_\infty}$.);
- And in degree 8, the volume form belonging to
$$\mathfrak{l}(k) = \begin{cases} 1 \text{ if } k = 2 \text{ or } k = 3 \\ 0 \text{ otherwise} \end{cases}$$
and $\mathfrak{x} = \{\{2, 3\}\} \in \mathfrak{X}_{0,0,\mathfrak{l}}$.

In the case $\mathbb{K} = \mathbb{Q}(i)$, this can be compared with the computation by R. Staffeldt ([25, Theorem IV.1.4.] combined with the Borel–Serre duality theorem [3, Theorem 11.4.1.]). It turns out that in this case all cohomology classes of $\mathrm{SL}_3(\mathbb{Z}[i])$ can be generated by Eisenstein series starting from the constant representation or by the constant representation itself. In particular, there are no harmonic cusp forms modulo $\mathrm{SL}_3(\mathbb{Z})$.

7.3 Homotopy type of a poset of partitions

As the main combinatorial tool in our computation of $H_c^*(\mathcal{G}, \mathbb{C})_{\mathcal{I}}$ for GL_n over imaginary quadratic fields, we use the description of the homotopy type of a partially ordered set of partitions.

In the following, we shall write ‘poset’ for ‘partially ordered set’. Let $\boldsymbol{B}X$ be the classifying space of the poset X. Notions from homotopy theory applied to posets or morphisms of posets will have the meaning of these notions, applied to the classifying space of the poset or morphism of posets. We will freely use the basic techniques for investigating the homotopy of the classifying space of a category (cf. [19] or the textbook [24]).

By an ordered partition of an integer n, we mean a tuple $(M, x_0, \ldots, x_M)$, where M is the number of intervals in the partition and $0 = x_0 < x_1 <$

$\cdots < x_M = n$ are the vertices of these intervals. We will say that a partition $(M, x_0, \ldots, x_M)$ is finer than or equal to $(N, y_0, \ldots, y_N)$, and write

$$(M, x_0, \ldots, x_M) \trianglelefteq (N, y_0, \ldots, y_N),$$

if $\{y_0, \ldots, y_N\} \subseteq \{x_0, \ldots, x_M\}$.

Consider a finite set $\mathfrak{S}$ and a function $F\colon \mathfrak{S} \to \{1, 2, \ldots\}$. Let $P_{n,\mathfrak{S},F}$ be the set of pairs $(f, (M, x_1, \ldots, x_M))$, where $(M, x_0, \ldots, x_M)$ is a partition of n and $f\colon \mathfrak{S} \to \{1, \ldots, M\}$ such that $x_{f(s)} - x_{f(s)-1} \geq F(s)$ for $s \in \mathfrak{S}$. In other words, elements of $P_{n,\mathfrak{S},F}$ are ordered partitions of n in which for each element $s \in \mathfrak{S}$ an interval of length $\geq F(s)$ is marked. The intervals associated to different elements of $\mathfrak{S}$ are not supposed to be different.

There is a partial order $\trianglelefteq$ on $P_{n,\mathfrak{S},F}$ for which

$$(f, (M, x_1, \ldots, x_M)) \trianglelefteq (g, (N, y_1, \ldots, y_N))$$

if and only if $(M, x_0, \ldots, x_M) \trianglelefteq (N, y_0, \ldots, y_N)$ and $y_{f(s)-1} \leq x_{f(s)-1} < x_{f(s)} \leq y_{f(s)}$ for $s \in \mathfrak{S}$. In other words, the partition $(M, x_0, \ldots, x_M)$ has to be finer than $(N, y_0, \ldots, y_N)$ and the interval in $(M, x_0, \ldots, x_M)$ associated to s by f must be contained in the interval in $(N, y_0, \ldots, y_N)$ associated to s by g.

If $n < \max\limits_{s \in \mathfrak{S}} F(s)$, the poset $P_{n,\mathfrak{S},F}$ is empty. Otherwise, it is contractible since it has a final object $(\mathbf{1}, (1, 0, n))$, where $\mathbf{1}$ is the constant function $s \to 1$ on $\mathfrak{S}$. Let

$$\tilde{P}_{n,\mathfrak{S},F} = p_{n,\mathfrak{S},F} - \{(\mathbf{1}, (1, 0, n))\}.$$

We will investigate the homotopy type of $\tilde{P}_{n,\mathfrak{S},F}$. It will turn out that it is a wedge of spheres. Before formulating our result, we have to define the index sets over which the wedge is taken. For $1 \leq k \leq \#(\mathfrak{S})$ and $\mathfrak{e} \in \{0, 1\}$, let $M_{n,\mathfrak{S},F,\mathfrak{e},k}$ be the set of ordered k-tuples $(\mathfrak{S}_1, \ldots, \mathfrak{S}_k)$ of nonempty mutually disjoint subsets of $\mathfrak{S}$ such that $\mathfrak{S} = \bigcup_{l=1}^{k} \mathfrak{S}_k$ and $n = \mathfrak{e} + \sum_{l=1}^{k} \max\limits_{s \in \mathfrak{S}_l} F(s)$.

Proposition 7.6. *If* $n = \max\limits_{s \in \mathfrak{S}} F(s)$, $\tilde{P}_{n,\mathfrak{S},F}$ *is empty. Let* $n > \max\limits_{s \in \mathfrak{S}} F(s)$. *Fix the basepoint* $✠ = \left(\mathbf{1}, (2, 0, \max\limits_{s \in \mathfrak{S}} F(s), n)\right)$ *of the poset* $\tilde{P}_{n,\mathfrak{S},F}$. *We have a homotopy equivalence of pointed spaces*

$$\phi_{n,\mathfrak{S},F}\colon \left(\boldsymbol{B}\tilde{P}_{n,\mathfrak{S},F}\right) \cong \bigvee_{\mathfrak{e}=0}^{1} \bigvee_{k=1}^{\#\mathfrak{S}} \bigvee_{M_{n,\mathfrak{S},F,\mathfrak{e},k}} S^{k+\mathfrak{e}-2}, \tag{7.21}$$

where S^l *is the pointed* l*-sphere (a set of two points if* $l = 0$*). It is assumed that the wedge over an empty index set is a contractible space.*

Moreover, if $\mathfrak{s} = (\mathfrak{S}_1, \ldots, \mathfrak{S}_k) \in M_{n,\mathfrak{S},F,0,k}$, *then the reduced cohomology class of its factor in (7.21) is given by the unrefinable chain of length* $k - 1$

$$x_1(\mathfrak{s}) \triangleleft x_2(\mathfrak{s}) \triangleleft \cdots \triangleleft x_{k-1}(\mathfrak{s}),$$

where

$$x_l(\mathfrak{s}) = \left(f_l^{\mathfrak{s}}, \left(k-l, \sum_{j=1}^{l} \#\mathfrak{S}_j, \sum_{j=1}^{l+1} \#\mathfrak{S}_j, \ldots, \sum_{j=1}^{k} \#\mathfrak{S}_j = n\right)\right)$$

and

$$f_l(s) = \begin{cases} 1 & \textit{if } s \in \bigcup_{i=1}^{l} \mathfrak{S}_i \\ k+1 & \textit{if } s \in \mathfrak{S}_{l+k} \textit{ with } k > 0. \end{cases}$$

Similarly, if $(\mathfrak{S}_1, \ldots, \mathfrak{S}_k) \in M_{n,\mathfrak{S},F,1,k}$, *then any of the following unrefinable chains of length* k *is a representative for the reduced cohomology class defined by the corresponding factor in the wedge (7.21). Take* $1 \leq m \leq k+1$, *define*

$$f_l^{(m)}(s) = \begin{cases} 1 & \textit{if } l < m \textit{ and } s \in \bigcup_{i=1}^{l} \mathfrak{S}_i \\ 2 & \textit{if } l \geq m \textit{ and } s \in \bigcup_{i=1}^{l} \mathfrak{S}_i \\ k+1 & \textit{if } s \in \mathfrak{S}_{k+l} \textit{ with } k > 0 \textit{ and } k+l < m \\ k+2 & \textit{if } s \in \mathfrak{S}_{k+l} \textit{ with } k > 0 \textit{ and } k+l \geq m \end{cases}$$

for $1 \leq l \leq k$ *and consider the chain*

$$x_1^{(m)} \lhd x_2 \lhd \cdots \lhd x_k^{(m)}$$

with

$$x_l^{(m)} = \Bigg(f_l^{(m)}, \Bigg(k+2-l, \sum_{i=1}^{l} \#\mathfrak{S}_i, \ldots \sum_{i=1}^{m-1} \#\mathfrak{S}_i, 1 + \sum_{i=1}^{m-1} \#\mathfrak{S}_i, 1 + \sum_{i=1}^{m} \#\mathfrak{S}_i, \ldots, n\Bigg)\Bigg)$$

if $l < m$ *and*

$$x_l^{(m)} = \left(f_l^{(m)}, \left(k+2-l, 1 + \sum_{i=1}^{m-1} \#\mathfrak{S}_i, 1 + \sum_{i=1}^{m} \#\mathfrak{S}_i, \ldots, n\right)\right)$$

otherwise.

Proof. It is clear that $\tilde{P}_{n,\mathfrak{S},F}$ is empty if $n = \max_{s\in\mathfrak{S}} F(s)$. If $n = 1 + \max_{s\in\mathfrak{S}} F(s)$, it is easy to see that $\tilde{P}_{n,\mathfrak{S},F}$ consists of two points without relation, and the theorem follows.

Let $n > 1 + \max_{s\in\mathfrak{S}} F(s)$. Let A be the poset of all elements $(f, (M, x_1, \ldots, x_M)) \in \tilde{P}_{n,\mathfrak{S},F}$ which satisfy one of the following two conditions:

- $f^{-1}(1)$ is empty;
- or $x_1 > \max_{s\in f^{-1}(1)} F(s)$.

Since $n > 1 + \max_{s \in \mathfrak{S}} F(s)$, $P_{n-1,\mathfrak{S},F}$ is contractible. We have an embedding

$$i_1 \colon P_{n-1,\mathfrak{S},F} \to A$$

defined by

$$\begin{aligned} & i_1\left((g,(N,y_1,\ldots,y_N))\right) \\ & \quad = \left(g+\mathbf{1},(N+1,0,1=y_1+1,y_2+1,\ldots,n=y_N+1)\right), \end{aligned}$$

where $g + \mathbf{1}$ is the function $s \to g(s) + 1$ on $\mathfrak{S}$. We also have a retraction for i_1

$$r_1 \colon A \to P_{n-1,\mathfrak{S},F}$$

which is defined by

$$\begin{aligned} & r_1\left((f,(M,x_1,\ldots,x_M))\right) \\ & \quad = \begin{cases} (f-\mathbf{1},(M-1,0=x_1-1,x_2-1,\ldots,n-1=x_M-1)) & \text{if } x_1 = 1 \\ (f,(M,0,x_1-1,\ldots,x_M-1=n-1)) & \text{if } x_1 > 1. \end{cases} \end{aligned}$$

Since $i_1 r_1\left(f,(M,x_1,\ldots,x_M)\right) \trianglelefteq \left(f,(M,x_1,\ldots,x_M)\right)$, $P_{n-1,\mathfrak{S},F}$ is a deformation retract of A, hence A is contractible.

Let $B \subset \tilde{P}_{n,\mathfrak{S},F}$ be the poset of all $(f,(M,x_1,\ldots,x_M))$ which satisfy at least one of the following two conditions:

- $(f,(M,x_1,\ldots,x_M)) \in A$;
- or $M > 2$.

We have the obvious inclusion $i_2 \colon A \to B$ and a retraction $r_2 \colon B \to A$ which is defined as follows. If $(f,(M,x_1,\ldots,x_M)) \in A$, we put

$$r_2((f,(M,x_1,\ldots,x_M))) = (f,(M,x_1,\ldots,x_M))\,.$$

If $(f,(M,x_1,\ldots,x_M)) \in B - A$, we define a function

$$h \colon \mathfrak{S} \to 1,\ldots,M-1$$

by

$$h(s) = \begin{cases} 1 & \text{if } f(s) = 1 \\ f(s)-1 & \text{if } f(s) > 1 \end{cases} \tag{7.22}$$

and put

$$r_2((f,(M,x_1,\ldots,x_M))) = (h,(M-1,x_0,x_2,\ldots,x_n))\,. \tag{7.23}$$

It is easy to see that r_2 is a morphism of posets, that $r_2 i_2 = \mathrm{Id}$, and that $i_2 r_2\left(f,(M,x_1,\ldots,x_M)\right) \trianglerighteq \left(f,(M,x_1,\ldots,x_M)\right)$. Therefore, A is a deformation retract of B, and B is contractible.

For $x \in \tilde{P}_{n,\mathfrak{S},F} - B$, let $B_-(x)$ be the poset of all $y \in B$ with $y \triangleleft x$. The set of all x for which $B_-(x)$ is empty can be identified with $M_{n,\mathfrak{S},F,0,2}$. Since no element of $\tilde{P}_{n,\mathfrak{S},F}$ can be coarser than x, the fact that B is contractible gives us a homotopy equivalence

$$\boldsymbol{B}\tilde{P}_{n,\mathfrak{S},F} \cong \bigvee_{\substack{x\in\tilde{P}_{n,\mathfrak{S},F}-B\\ B_-(x)\neq\emptyset}} \Sigma(\boldsymbol{B}B_-(x)) \vee \bigvee_{M_{n,\mathfrak{S},F,0,2}} S^0, \tag{7.24}$$

where Σ is the suspension functor.

Let us first assume that $\mathfrak{S}$ consists of a single element s. The assumption made at the beginning of the proof means that $n > F(s) + 1$. Then all sets $M_{n,\mathfrak{S},F,\mathfrak{e},k}$ are empty, and we have to show that $\tilde{P}_{n,\mathfrak{S},F}$ is contractible. The only factor in (7.24) is $B_-(✠)$, which has an initial object

$$(\mathbf{1}, (n - F(s), 0, F(s), F(s) + 1, \ldots, n)) .$$

This completes the proof of the proposition if $\mathfrak{S}$ has only one element.

Now we assume by induction that the proposition has been verified for all subsets of $\mathfrak{S}$. As above, $B_-(✠)$ has an initial object and is contractible. The other elements of $\tilde{P}_{n,\mathfrak{S},F} - B$ for which $B_-(x)$ is not empty are of the form

$$x = \left(\begin{pmatrix} 1 \text{ on } T \\ 2 \text{ on } \mathfrak{S} - T \end{pmatrix}, \left(2, 0, \max_{s\in T} F(s), n\right)\right),$$

where T is a nonempty subset of $\mathfrak{S}$ such that $\max_{s\in T} F(s) + \max_{s\notin T} F(s) \leq n$. For such x we have

$$B_-(x) = \tilde{P}_{n-\max_{s\in T} F(s),\mathfrak{S}-T,F}.$$

Then a combination of (7.24) with the induction assumption gives us

$$\begin{aligned}\boldsymbol{B}\tilde{P}_{n,\mathfrak{S},F} &\cong \bigvee_{\substack{T\subset\mathfrak{S}\\ T\neq\emptyset\\ \max_{s\in T}F(s)+\max_{s\notin T}F(s)<n}} \Sigma(\boldsymbol{B}\tilde{P}_{n-\max_{s\in T}F(s),\mathfrak{S}-T,F}) \vee \bigvee_{M_{n,\mathfrak{S},F,0,2}} S^0 \\ &\cong \left(\bigvee_{\substack{T\subset\mathfrak{S}\\ T\neq\emptyset\\ \max_{s\in T}F(s)+\max_{s\notin T}F(s)<n}} \bigvee_{\mathfrak{e}=0}^{1} \bigvee_{k=1}^{\#(\mathfrak{S}-T)} \bigvee_{M_{n-\max_{s\in T}F(s),\mathfrak{S}-T,F,\mathfrak{e},k}} S^{k+\mathfrak{e}-1}\right) \\ &\qquad \vee \bigvee_{M_{n,\mathfrak{S},F,0,2}} S^0.\end{aligned}$$

Since the maps

$$\begin{aligned} M_{n-\max_{s\in T}F(s),\mathfrak{S}-T,F,\mathfrak{e},k} &\to M_{n,\mathfrak{S},F,\mathfrak{e},k+1} \\ (\mathfrak{S}_1,\ldots,\mathfrak{S}_{k-1}) &\to (T,\mathfrak{S}_1,\ldots,\mathfrak{S}_{k-1}) \end{aligned}$$

define an isomorphism

$$\bigcup_{\substack{T\subset\mathfrak{S}\\ T\neq\emptyset\\ \max_{s\in T}F(s)+\max_{s\notin T}F(s)<n}} M_{n-\max_{s\in T}F(s),\mathfrak{S}-T,F,\mathfrak{e},k}\cong M_{n,\mathfrak{S},F,\mathfrak{e},k+1},$$

this completes the induction argument. The explicit formula for the reduced cohomology classes defined by the individual factors in the wedge (7.21) can easily be verified by induction. □

We now want to explain how one can translate homology computations for certain posets into assertions about the homology of functors from $\mathfrak{P}$ to abelian groups. Let

$$p\colon(X,\lhd)\to(\mathfrak{P},\subset)$$

be a morphism of posets such that $p^{-1}(\mathcal{G})$ is empty and such that

a. for $\mathcal{G}\supset\mathcal{P}\supseteq\mathcal{Q}$ and $x\in p^{-1}(\mathcal{Q})$, there is a unique $y\in p^{-1}(\mathcal{P})$ with $y\unrhd x$.

We define a functor $\boldsymbol{J}^{\bullet}_{X,p}$ by

$$\boldsymbol{J}^{\mathcal{P}}_{X,p}=\begin{cases}\bigoplus_{x\in p^{-1}(\mathcal{P})}\mathbb{C}x \text{ if } \mathcal{P}\subset\mathcal{G}\\ \mathbb{C}\text{if } \mathcal{P}=\mathcal{G}\end{cases}\tag{7.25}$$

and

$$\boldsymbol{J}^{\mathcal{Q}\subseteq\mathcal{P}}_{X,p}(x)=\sum_{\substack{y\in p^{-1}(\mathcal{Q})\\ y\lhd x}} y$$

for $x\in p^{-1}(\mathcal{P})$ with $\mathcal{P}\subset\mathcal{G}$ and

$$\boldsymbol{J}^{\mathcal{Q}\subseteq\mathcal{G}}_{X,p}1=\sum_{y\in p^{-1}(\mathcal{Q})} y.$$

Proposition 7.7. *Assume condition a. above and assume moreover the condition*

b. If $x_1,\dots,x_k\in X$ such that $p(x_i)$ is a maximal parabolic subgroup for $1\leq i\leq k$, then there is at most one $y\in p^{-1}(p(x_1)\cap\cdots\cap p(x_k))$ with $y\unlhd x_i$ for all $1\leq i\leq k$.

Under these circumstances, we have a canonical isomorphism

$$H^*(C^*(\boldsymbol{J}^{\bullet}_{X,p}))[1]\cong\tilde{H}^*(\boldsymbol{B}X).\tag{7.26}$$

Moreover, let us assume that the differential on $C^*(\boldsymbol{J}^{\bullet}_{X,p})$ was defined using the order $\prec$ on Δ_o. Let $\xi=(x_1\lhd x_2\lhd\dots x_k)$ be an unrefinable chain in X, defining a reduced cohomology class $[\xi]$ in degree $k-1$ on $\boldsymbol{B}X$. Then x_1 cannot be refined, hence it defines a cohomology class in degree k for $\boldsymbol{J}_{X,p}$. Let α_i be the unique element of $\Delta_o^{p(x_{i+1})}-\Delta_o^{p(x_i)}$ if $i<k$ and the unique element of $\Delta_o-\Delta_o^{p(x_k)}$ if $i=k$. Further, let $\varepsilon\in\{1;-1\}$ be the orientation with respect to $\prec$ of $\alpha_1,\dots,\alpha_k$. Then (7.26) maps $[\xi]$ to εx_k.

Proof. Let us define a simplicial complex (Y, Σ) as follows. The set of vertices Y is the set of $x \in X$ such that $p(x)$ is a standard maximal parabolic subalgebra. A k-tuple $(x_1, \ldots, x_k)$ of vertices belongs to the set Σ of simplices if there exists an $x \in X$ with $x \trianglelefteq x_k$ for all k. By conditions a. and b. above, the reduced chain complex for computing the cohomology of (Y, Σ) is $C^*(\boldsymbol{J}^\bullet_{X,p})[1]$. The proposition now follows from the well-known fact that the barycentric subdivision of a simplicial complex is the nerve of its poset of simplices, which in the case of (Y, Σ) is X. □

7.4 Proof of Theorem 7.5

We now prove the explicit formulas for the Eisenstein cohomology which we announced earlier. We are considering the group $\mathcal{G} = \mathrm{res}^{\mathbb{K}}_{\mathbb{Q}} \mathcal{GL}_n$ for an imaginary quadratic field $\mathbb{K}$.

To prove Theorem 7.5, consider $\mathfrak{l} \in \mathfrak{L}_n(\mathbb{K})$. If $\mathfrak{l}(n) = 1$, then $\mathfrak{Y}_{\mathfrak{l},\mathcal{P}}$ is empty unless $\mathcal{P} = \mathcal{G}$, in which case it has precisely one element. It follows that $C^*(\mathfrak{I}^\bullet_{\mathfrak{l}})$ has a one-dimensional cohomology group in dimension zero, and no other cohomology. Also, $\mathfrak{X}_{N,\mathfrak{e},\mathfrak{l}}$ is empty unless $N = 1$ and $\mathfrak{e} = 0$, in which case it consists of a single element. This proves the theorem for those $\mathfrak{l}$ with $\mathfrak{l}(n) = 1$. The case $\mathfrak{l}(n) > 1$ is excluded by the condition (7.10). Therefore we suppose for the remaining part of this proof that $\mathfrak{l}(n) = 0$.

We define the set $\mathfrak{S}_{\mathfrak{l}}$ by

$$\mathfrak{S}_{\mathfrak{l}} = \{(k, l) \,|2 \leq k \leq n, 1 \leq l \leq \mathfrak{l}(k)\,\}$$

and define the function $F\colon \mathfrak{S}_{\mathfrak{l}} \to \{1, 2, \ldots\}$ by $F\left((k, l)\right) = k$. Since $\mathfrak{l}(n) = 0$, the poset $\tilde{P}_{n,\mathfrak{S}_{\mathfrak{l}},F}$ defined in the last subsection is not empty. We have the map

$$p_{\mathfrak{l}}\colon \tilde{P}_{n,\mathfrak{S}_{\mathfrak{l}},F} \to \mathfrak{P}$$

from $\tilde{P}_{n,\mathfrak{S}_{\mathfrak{l}},F}$ to the poset $\mathfrak{P}$ of standard parabolic subgroups which associates to the tuple $(f, (M, i_1, \ldots, i_M))$ the parabolic subgroup of type $0 < i_1 < \cdots < i_M = n$, i.e., the stabilizer of the standard flag of subspaces of succesive dimension i_k. The formula (7.25) now defines us a functor $\boldsymbol{J}_{\tilde{P}_{n,\mathfrak{S}_{\mathfrak{l}},F},p_{\mathfrak{l}}}$ from $\mathfrak{P}$ to vector spaces whose homology is known by Proposition 7.6 and Proposition 7.7. We will express $\boldsymbol{I}^{\mathcal{P}}_{\mathfrak{l}}$ as an "antisymmetrization" of $\boldsymbol{J}_{\tilde{P}_{n,\mathfrak{S}_{\mathfrak{l}},F},p_{\mathfrak{l}}}$.

The product of the symmetric groups $\prod_{k=2}^n S_{\mathfrak{l}(k)}$ acts on the set $\mathfrak{S}_{\mathfrak{l}}$ by permutation of the second entry of the pairs (k, l) which form $\mathfrak{S}_{\mathfrak{l}}$. This permutation leaves F invariant, therefore it extends to an action of the group $\prod_{k=2}^n S_{\mathfrak{l}(k)}$ on the poset $\tilde{P}_{n,\mathfrak{S}_{\mathfrak{l}},F}$. This action leaves $p_{\mathfrak{l}}$ invariant, therefore it extends to an action of $\prod_{k=2}^n S_{\mathfrak{l}(k)}$ on the functor $\boldsymbol{J}_{\tilde{P}_{n,\mathfrak{S}_{\mathfrak{l}},F},p_{\mathfrak{l}}}$. We want to consider the antisymmetrization of $\boldsymbol{J}_{\tilde{P}_{n,\mathfrak{S}_{\mathfrak{l}},F},p_{\mathfrak{l}}}$ with respect to this action.

For each parabolic subgroup $\mathcal{P}$, we have an injective map of sets

$$\mathfrak{i}\colon \mathfrak{Y}_{\mathfrak{l},\mathcal{P}} \to p_{\mathfrak{l}}^{-1}(\mathcal{P})$$

which maps the element $\mathfrak{y} \in \mathfrak{Y}_{\mathfrak{l},\mathcal{P}}$ to the element

$$(\mathfrak{y}, (K, i_1, \dots, i_K)) \in \mathfrak{S}_{\mathfrak{l}}.$$

By condition (7.11), this element really belongs to $\mathfrak{S}_{\mathfrak{l}}$. Consider an element $(f, (K, i_1, \dots, i_K))$ of $p^{-1}(\mathcal{P})$. If there exists a k and if $1 \le l_1 < l_2 \le \mathfrak{l}(k)$, then exchanging (k, l_1) and (k, l_2) is an odd element of $\prod_{k=2}^{n} S_{\mathfrak{l}(k)}$ which leaves $(f, (K, i_1, \dots, i_K))$ fixed. Therefore, the image of $(f, (K, i_1, \dots, i_K))$ in the anisymmetrization of $\boldsymbol{J}_{\tilde{P}_{n,\mathfrak{S}_{\mathfrak{l}},F,p_{\mathfrak{l}}}}$ vanishes. Otherwise, the $\prod_{k=2}^{n} S_{\mathfrak{l}(k)}$-orbit of $(f, (K, i_1, \dots, i_K))$ contains an element in the image of $\mathfrak{i}$, which is unique by (7.12). This identifies $\boldsymbol{I}_{\mathfrak{l}}^{\bullet}$ with the antisymmetrization of $\boldsymbol{J}_{\tilde{P}_{n,\mathfrak{S}_{\mathfrak{l}},F,p_{\mathfrak{l}}}}$.

By Proposition 7.6 and Proposition 7.7, the cohomology of $\boldsymbol{J}_{\tilde{P}_{n,\mathfrak{S}_{\mathfrak{l}},F,p_{\mathfrak{l}}}}$ is a graded vector space with a basis given by the sets $M_{n,\mathfrak{S}_{\mathfrak{l}},F,\mathfrak{e},k}$. A permutation $\mathfrak{p}$ in $\prod_{k=2}^{n} S_{\mathfrak{l}(k)}$ acts on these sets by

$$\pi\colon (\mathfrak{S}_1, \dots, \mathfrak{S}_k) \to (\pi(\mathfrak{S}_1), \dots, \pi(\mathfrak{S}_k)),$$

and this action commutes with the action on the cohomology of $\boldsymbol{J}_{\tilde{P}_{n,\mathfrak{S}_{\mathfrak{l}},F,p_{\mathfrak{l}}}}$. We have the map

$$\mathfrak{j}\colon \mathfrak{X}_{N,\mathfrak{e},\mathfrak{l}} \to M_{n,\mathfrak{S}_{\mathfrak{l}},F,\mathfrak{e},N+1}$$

which maps the collection $X_0, \dots, X_N$ of subsets of $\{2, \dots, n\}$ to the disjoint partition $\mathfrak{S}_{\mathfrak{l}} = \bigcup_{j=1}^{N+1} \mathfrak{S}_j$, where

$$\mathfrak{S}_j = \Big\{(k,l) \in \mathfrak{S}_{\mathfrak{l}} \;\Big|\; k \in X_{j+1}, \text{ and there are precisely } l-1 \text{ elements } i \text{ with } 0 \le i < j \text{ and } k \in X_{i+1}\Big\}.$$

If $(S_1, \dots, S_{N+1}) \in M_{n,\mathfrak{S}_{\mathfrak{l}},F,\mathfrak{e},N+1}$ and if there exists $2 \le k < n$ and $1 \le l_1 < l_2 \le \mathfrak{l}(k)$, then exchanging (k, l_1) and (k, l_2) is an odd element of $\prod_{k=2}^{n} S_{\mathfrak{l}(k)}$ which leaves $(S_1, \dots, S_{N+1})$ fixed. Therefore, the image of the generator belonging to $(S_1, \dots, S_{N+1})$ vanishes in the antisymmetrization of the cohomology of $\boldsymbol{J}_{\tilde{P}_{n,\mathfrak{S}_{\mathfrak{l}},F,p_{\mathfrak{l}}}}$. Otherwise, the $\prod_{k=2}^{n} S_{\mathfrak{l}(k)}$-orbit of $(S_1, \dots, S_{N+1})$ contains a unique element in the image of $\mathfrak{j}$.

We have identified $\boldsymbol{I}_{\mathfrak{l}}^{\bullet}$ with the antisymmetrization of $\boldsymbol{J}_{\tilde{P}_{n,\mathfrak{S}_{\mathfrak{l}},F,p_{\mathfrak{l}}}}$ and the right-hand side of (7.18) with the antisymmetrization of the homology of $\boldsymbol{J}_{\tilde{P}_{n,\mathfrak{S}_{\mathfrak{l}},F,p_{\mathfrak{l}}}}$. This proves (7.18). By the remarks made before the formulation of Theorem 7.5, this also completes the computation of the spherical subspace of $H_c^*(\mathcal{G}, \mathbb{C})_{\mathcal{I}}$.

To get a result about the nonspherical vectors in the cohomology, we have to investigate the cohomology of the functor

$$\tilde{\boldsymbol{J}}^{\mathcal{P}}_{n,\mathfrak{S},F} = \boldsymbol{J}_{\tilde{P}_{n,\mathfrak{S},F},p_{\mathfrak{l}}} \otimes C^\infty(\mathcal{P}(\mathbb{A}_f)\backslash\mathcal{G}(\mathbb{A}_f)),$$

where $\mathfrak{S}$ is a finite set and F is a function from $\mathfrak{S}$ to integers. Since both the formulation and the proof of the result are straightforward but quite unpleasant,the result will be formulated precisely but the proof will only be sketched. Let $\mathfrak{Q}_k(\mathfrak{S})$ be the set of partitions

$$\mathfrak{s}\colon \mathfrak{S} = \bigcup_{l=1}^{k} \mathfrak{S}_l$$

into k disjoint pieces. For $\mathfrak{s} \in \mathfrak{Q}_k(\mathfrak{S})$, let $\mathfrak{A}_{\mathfrak{s},\mathfrak{S},F}$ be the set of pairs $(\mathcal{P},\mathfrak{f})$ with the following properties:

- $\mathcal{P}$ is a standard parabolic subgroup, stabilizing the standard flag of subspaces of dimensions

$$0 = i_0^{\mathcal{P}} < i_1^{\mathcal{P}} < \cdots < i^{\mathcal{P}}_{\dim \mathfrak{a}^{\mathcal{G}}_{\mathcal{P}}} = n.$$

- $\mathfrak{f}$ is a monotonous map from $\{1,\ldots,k\}$ to $\{1,\ldots,\dim \mathfrak{a}^{\mathcal{G}}_{\mathcal{P}}\}$ such that

$$i_{\mathfrak{f}(j)} - i_{\mathfrak{f}(j)-1} = \max_{s\in\mathfrak{S}_j} F(s).$$

- If $j \in \{1,\ldots,\dim \mathfrak{a}^{\mathcal{G}}_{\mathcal{P}}\} - \mathfrak{f}(\{1,\ldots,k\})$, then $i_j - i_{j-1} = 1$.

Note that the rank of $\mathcal{P}$ is uniquely determined; it is equal to

$$\mathfrak{d}(\mathfrak{s}) = \dim \mathfrak{a}^{\mathcal{G}}_{\mathcal{P}} = k + n - \sum_{j=1}^{k} \#(\mathfrak{S}_k).$$

For $(\mathcal{P},\mathfrak{f}) \in \mathfrak{A}_{\mathfrak{s},\mathfrak{S},F}$, let $x_{\mathcal{P},\mathfrak{f}} \in P_{n,\mathfrak{S},F}$ be the element

$$\left(\mathfrak{f}^{\spadesuit}, (\dim \mathfrak{a}^{\mathcal{G}}_{\mathcal{P}}, i_1, \ldots, i_{\dim \mathfrak{a}^{\mathcal{G}}_{\mathcal{P}}})\right),$$

where $\mathfrak{f}^{\spadesuit}$ is equal to $\mathfrak{f}(j)$ on $\mathfrak{S}_j$. This is a minimal element of $P_{n,\mathfrak{S},F}$ which lies over $\mathcal{P}$. We get a homomorphism

$$\mathfrak{a}_{\mathfrak{s}}\colon \bigoplus_{(\mathcal{P},\mathfrak{f})\in\mathfrak{A}_{\mathfrak{s}}} C^\infty(\mathcal{P}(\mathbb{A}_f)\backslash\mathcal{G}(\mathbb{A}_f)) \to H^{\mathfrak{d}(\mathfrak{s})}\left(C^*(\tilde{\boldsymbol{I}}^{\bullet}_{n,\mathfrak{S},F})\right)$$
$$(f_{\mathcal{P},\mathfrak{f}})_{(\mathcal{P},\mathfrak{f}\in\mathfrak{A}_{\mathfrak{s}})} \to \sum_{(\mathcal{P},\mathfrak{f})\in\mathfrak{A}_{\mathfrak{s}}} f_{\mathcal{P},\mathfrak{f}} \otimes x_{\mathcal{P},\mathfrak{f}}.$$

If $\mathcal{Q}$ is a parabolic subgroup of rank $> \mathfrak{d}(\mathfrak{s})$, let $\mathfrak{B}^{\dagger}_{\mathcal{Q},\mathfrak{s}}$ be the set of all pairs $\left((\mathcal{P},\mathfrak{f}),(\tilde{\mathcal{P}},\tilde{\mathfrak{f}})\right)$ with the following properties:

- We have $(\mathcal{P},\mathfrak{f}),(\tilde{\mathcal{P}},\tilde{\mathfrak{f}}) \in \mathfrak{A}_{\mathfrak{s}}$ and $\mathcal{Q} \supset \mathcal{P}$, $\mathcal{Q} \supset \tilde{\mathcal{P}}$.

- Let $0 = i_0^{\mathcal{Q}} < i_1^{\mathcal{Q}} < \cdots < i_{\dim \mathfrak{a}_{\mathcal{Q}}^{\mathcal{G}}}^{\mathcal{Q}} = n$ be the dimension of the spaces in the standard flag defining $\mathcal{Q}$. For each $j \in \{1, \ldots, k\}$, there exists an l with
$$i_{l-1}^{\mathcal{Q}} < i_{\mathfrak{f}(j)}^{\mathcal{P}} \leq i_l^{\mathcal{Q}}$$
and
$$i_{l-1}^{\mathcal{Q}} < i_{\tilde{\mathfrak{f}}(j)}^{\tilde{\mathcal{P}}} \leq i_l^{\mathcal{Q}}.$$
In other words, the intervals $[i_{\mathfrak{f}(j)-1}^{\mathcal{P}} + 1; i_{\mathfrak{f}(j)}^{\mathcal{P}}]$ and $[i_{\tilde{\mathfrak{f}}(j)-1}^{\tilde{\mathcal{P}}} + 1; i_{\tilde{\mathfrak{f}}(j)}^{\tilde{\mathcal{P}}}]$ are contained in the same interval of the partition $i_m^{\mathcal{Q}}$.
- We have
$$\sum_{\substack{1 \leq j \leq k \\ i_{l-1}^{\mathcal{Q}} < i_{\mathfrak{f}(j)}^{\mathcal{P}} \leq i_l^{\mathcal{Q}}}} \#\mathfrak{S}_j = i_l^{\mathcal{Q}} - i_{l-1}^{\mathcal{Q}} - 1.$$
An empty sum is supposed to be zero. Note that by the previous assumption, the sum on the left-hand side of the inequality is also equal to
$$\sum_{\substack{1 \leq j \leq k \\ i_{l-1}^{\mathcal{Q}} < i_{\tilde{\mathfrak{f}}(j)}^{\tilde{\mathcal{P}}} \leq i_l^{\mathcal{Q}}}} \#\mathfrak{S}_j.$$

We have the homomorphism
$$\mathfrak{b}_{\mathfrak{s}}^{\dagger}\colon \bigoplus_{\mathcal{Q} \in \ \dim \mathfrak{a}_{\mathcal{Q}}^{\mathcal{G}} > \mathfrak{d}(\mathfrak{s})} \ \bigoplus_{\left((\mathcal{P},\mathfrak{f}),(\tilde{\mathcal{P}},\tilde{\mathfrak{f}})\right) \in \mathfrak{B}_{\mathcal{Q},\mathfrak{s}}^{\dagger}} \to \bigoplus_{(\mathcal{P},\mathfrak{f}) \in \mathfrak{A}_{\mathfrak{s}}} C^{\infty}(\mathcal{P}(\mathbb{A}_f) \backslash \mathcal{G}(\mathbb{A}_f))$$
which for $\left((\mathcal{P}, \mathfrak{f}), (\tilde{\mathcal{P}}, \tilde{\mathfrak{f}})\right)$ maps $f \in C^{\infty}(\mathcal{Q}(\mathbb{A}_f) \backslash \mathcal{G}(\mathbb{A}_f))$ to $f \otimes (\mathcal{P}, \mathfrak{f}) - f \otimes (\tilde{\mathcal{P}}, \tilde{\mathfrak{f}})$. Similarly, let $\mathfrak{C}_{\mathcal{Q},\mathfrak{s}}^{\dagger}$ be the set of all $(\mathcal{P}, \mathfrak{f}) \in \mathfrak{A}_{\mathfrak{s}}$ such that $\mathcal{P} \subset \mathcal{Q}$ and such that there exists an l with
$$i_l^{\mathcal{Q}} - i_{l-1}^{\mathcal{Q}} - 1 > \sum_{\substack{1 \leq j \leq k \\ i_{l-1}^{\mathcal{Q}} < i_{\mathfrak{f}(j)}^{\mathcal{P}} \leq i_l^{\mathcal{Q}}}} \#\mathfrak{S}_j.$$

Let $\mathfrak{c}_{\mathfrak{s}}^{\dagger}$ be the obvious map
$$\bigoplus_{\mathcal{Q} \in \ \dim \mathfrak{a}_{\mathcal{Q}}^{\mathcal{G}} > \mathfrak{d}(\mathfrak{s})} \ \bigoplus_{(\mathcal{P},\mathfrak{f}) \in \mathfrak{C}_{\mathcal{Q},\mathfrak{s}}^{\dagger}} C^{\infty}(\mathcal{Q}(\mathbb{A}_f) \backslash \mathcal{G}(\mathbb{A}_f)) \to \bigoplus_{(\mathcal{P},\mathfrak{f}) \in \mathfrak{A}_{\mathfrak{s}}} C^{\infty}(\mathcal{P}(\mathbb{A}_f) \backslash \mathcal{G}(\mathbb{A}_f)).$$

Theorem 7.8. *The kernel of $\tilde{a}_{\mathfrak{s}}$ is equal to the image of $\mathfrak{b}_{\mathfrak{s}}^{\dagger} \oplus \mathfrak{c}_{\mathfrak{s}}^{\dagger}$, and we have an isomorphism of $\mathcal{G}(\mathbb{A}_f)$-modules*
$$H^*\left(C^*(\tilde{\boldsymbol{J}}_{n,\mathfrak{S},F})\right) \cong \bigoplus_k \bigoplus_{\mathfrak{s} \in \mathfrak{Q}_k(\mathfrak{S})} \operatorname{coker}(\mathfrak{b}_{\mathfrak{s}}^{\dagger} \oplus \mathfrak{c}_{\mathfrak{s}}^{\dagger})[-\mathfrak{d}(\mathfrak{s})]. \tag{7.27}$$

To prove the theorem, one filters $\tilde{\mathfrak{I}}^{\mathcal{P}}_{n,\mathfrak{S},f}$ by the subspaces

$$\bigoplus_{\substack{\mathcal{R}\in\mathfrak{P}\\ \mathcal{R}\subseteq\mathcal{P}\\ \dim\mathfrak{a}^{\mathcal{G}}_{\mathcal{R}}\leq k}} C^\infty(\mathcal{R}(\mathbb{A}_f)\backslash\mathcal{G}(\mathbb{A}_f))\otimes \boldsymbol{J}_{\tilde{P}_{n,\mathfrak{S}_{\mathfrak{l}},F},p_{\mathfrak{l}}},$$

to define a similar filtration on the sources of $\mathfrak{a}_{\mathfrak{s}}$, $\mathfrak{b}^\dagger_{\mathfrak{s}}$ and $\mathfrak{c}^\dagger_{\mathfrak{s}}$, and to derive the theorem for the grading from Proposition 7.6 and Proposition 7.7.

To compute $H^*(\mathcal{G},\mathbb{C})_{\mathcal{I}}$, recall that $\boldsymbol{I}^{\mathcal{P}}_{\mathfrak{l}}$ is the antisymmetrization of $\tilde{\boldsymbol{J}}_{n,\mathfrak{S}_{\mathfrak{l}},F}$ with respect to the group $\prod_{k=2}^n S_{\mathfrak{l}(k)}$ and note that (7.27) identifies the action of this group on the cohomology of $\tilde{\boldsymbol{J}}_{n,\mathfrak{S}_{\mathfrak{l}},F}$ with the action on the right-hand side of (7.27) derived by permutation of the elements of the set $\mathfrak{Q}_k(\mathfrak{S}_{\mathfrak{l}})$. If therefore $\mathfrak{Q}^{\mathrm{mon}}_k(\mathfrak{S}_{\mathfrak{l}})$ is the set of all $\mathfrak{s}=(\mathfrak{S}_1,\ldots,\mathfrak{S}_k)\in\mathfrak{Q}_k(\mathfrak{S}_{\mathfrak{l}})$ such that, if $1\leq l_1<l_2<\mathfrak{l}(m)$ and $(m,l_1)\in\mathfrak{S}_{i_1}$ and $(m,l_2)\in\mathfrak{S}_{i_2}$ then $i_1<i_2$, we then get

Theorem 7.9. *We have a canonical isomorphism*

$$H^*(\mathcal{G},\mathbb{C})_{\mathcal{I}}\cong\bigoplus_{\mathfrak{l}\in\mathfrak{L}_n(\mathbb{K})}\bigoplus_{k}\bigoplus_{\mathfrak{s}\in\mathfrak{Q}^{\mathrm{mon}}_k(\mathfrak{S}_{\mathfrak{l}})}\operatorname{coker}(\mathfrak{b}^\dagger_{\mathfrak{s}}\oplus\mathfrak{c}^\dagger_{\mathfrak{s}})[-\mathfrak{d}(\mathfrak{s})-\deg\mathfrak{l}]. \qquad (7.28)$$

7.5 The case $\mathbf{SL}_{n}(\mathbb{Z})$

Here we consider the case $\mathcal{G}=\mathrm{SL}_n$. We want to explicitly compute the space of spherical vectors in $H^*(\mathcal{G},\mathbb{C})_{\mathcal{I}}$ and to compare the result with computations by C. Soulé and J. Schwermer for $n=3$ and by R. Lee and R. H. Szczarba for $n=4$.

We start with an explicit description of the spaces $\check{\boldsymbol{H}}(\mathcal{G})^{\mathcal{P}*\boldsymbol{K}_f}$. Recall that the minimal parabolic subgroup $\mathcal{P}_o$ is the stabilizer of a standard full flag $V_1\subset V_2\subset\cdots\subset V_n=\mathbb{K}^n$. Let $\mathcal{P}$ be the stabilizer of the subflag $V_{i_1}\subset V_{i_2}\subset\cdots\subset V_{i_K}$ for some sequence $0<i_1<i_2<\cdots<i_K=n$. Then

$$\mathcal{M}_{\mathcal{P}}=\prod_{l=1}^K \mathrm{SL}_{i_l-i_{l-1}}\,.$$

By Proposition 7.2, the cohomology of $\boldsymbol{X}^{(c)}_{\mathcal{M}_{\mathcal{P}}}$ is an exterior algebra which, for $1\leq l\leq K$, has the following generators:

$$\begin{array}{ll}\tilde{\lambda}^{(l)}_3,\tilde{\lambda}^{(l)}_5,\ldots,\tilde{\lambda}^{(l)}_n & \text{if } i_l-i_{l-1} \text{ is odd}\\ \tilde{\lambda}^{(l)}_3,\tilde{\lambda}^{(l)}_5,\ldots,\tilde{\lambda}^{(l)}_{n-1},\varepsilon^{(l)} & \text{if } i_l-i_{l-1} \text{ is even.}\end{array} \qquad (7.29)$$

The group

$$\pi_0\big(\mathrm{SO}(n,\mathbb{R})\cap\mathcal{L}_{\mathcal{P}}(\mathbb{R})\big)\cong\Big\{\sigma_1,\ldots,\sigma_K\in\{\pm1\}\;\Big|\;\prod_{l=1}^K\sigma_l=1\Big\}$$

acts on this cohomology algebra, and only the invariants will contribute to $\check{\boldsymbol{H}}(\mathcal{G})^{\mathcal{P}*\boldsymbol{K}_f}$. Using the fact that $\tilde{\lambda}_i^{(l)}$ is obtained by pull-back with respect to (7.3), one easily sees that the classes $\tilde{\lambda}_i^{(l)}$ are $\pi_0\big(\mathrm{SO}(n,\mathbb{R})\cap\mathcal{L}_{\mathcal{P}}(\mathbb{R})\big)$-invariant. However, conjugation by an element of $\boldsymbol{O}(n,\mathbb{R})-\mathrm{SO}(n,\mathbb{R})$ changes the orientation of the canonical n-dimensional real bundle on $\mathrm{SU}(n,\mathbb{R})/\mathrm{SO}(n,\mathbb{R})$, hence $\big(\sigma_i\big)_{i=1}^{K}\in\pi_0\big(\mathrm{SO}(n,\mathbb{R})\cap\mathcal{L}_{\mathcal{P}}(\mathbb{R})\big)$ maps $\varepsilon^{(l)}$ to $\sigma_l\varepsilon^{(l)}$. This means that a monomial μ in the generators (7.29) is $\pi_0\big(\mathrm{SO}(n,\mathbb{R})\cap\mathcal{L}_{\mathcal{P}}(\mathbb{R})\big)$-invariant if and only if one of the following cases occurs:

- μ contains no Euler class $\varepsilon^{(l)}$;
- or the numbers i_l-i_{l-1} are all even, and μ contains all Euler classes $\varepsilon^{(l)}$.

It follows that

$$\check{\boldsymbol{H}}(\mathcal{G})^{\mathcal{P}*\boldsymbol{K}_f}=\begin{cases}\boldsymbol{M}^{*\mathcal{P}}_{(n)} & n\text{ odd}\\ \boldsymbol{M}^{*\mathcal{P}}_{(n)}\oplus{}^{e}\boldsymbol{M}^{*\mathcal{P}}_{(n)} & n\text{ even,}\end{cases}\tag{7.30}$$

where

$$\boldsymbol{M}^{*\mathcal{P}}_{(n)}=\Big\{\text{monomials in the }\tilde{\lambda}_i^{(l)}\Big\}$$

and

$${}^{e}\boldsymbol{M}^{*\mathcal{P}}_{(n)}=\begin{cases}\prod_{l=1}^{K}\varepsilon^{(l)}\cdot\Big\{\text{monomials in the }\tilde{\lambda}_i^{(l)}\Big\} & \text{if all the numbers } i_l,\ 1\le l\le K,\text{ are even}\\ \{0\} & \text{otherwise.}\end{cases}$$

The explicit formulas for the restriction of cohomology classes in Proposition 7.2 show that, for n even, the decomposition (7.30) is functorial in $\mathcal{P}$.

We first give an explicit formula for the first summand in (7.30). Let

$$\mathrm{Odd}_{\le n}:=\begin{cases}\{3,\dots,n\} & \text{if } n \text{ is odd}\\ \{3,\dots,n-1\} & \text{if } n \text{ is even.}\end{cases}$$

Let $\mathfrak{L}_n(\mathbb{Q})$ be the set of functions

$$\mathfrak{l}\colon \mathrm{Odd}_{\le n}\to\{0,1,\dots\}$$

satisfying the condition

$$\sum_{j=1}^{\infty}\max\{k\in\mathrm{Odd}_{\le n}\,|\,\mathfrak{l}(k)\ge j\}\le n.\tag{7.31}$$

If the parabolic subgroup $\mathcal{P}$ corresponds to $0<i_1<\cdots<i_K=n$, let $\mathfrak{Y}_{\mathfrak{l},\mathcal{P}}$ be defined in the same way as in the case of imaginary quadratic fields, i.e., as the set of functions

$$\mathfrak{y}\colon\ \{(k,l)\,|\,k\in\mathrm{Odd}_{\le n},1\le l\le\mathfrak{l}(k)\}\to\{1,\dots,k\}$$

with the properties (7.11) and (7.12). For $\tilde{\mathcal{P}} \supseteq \mathcal{P}$, $\mathfrak{y} \in \mathfrak{Y}_{\mathfrak{l},\mathcal{P}}$ and $\tilde{\mathfrak{y}} \in \mathfrak{Y}_{\mathfrak{l},\tilde{\mathcal{P}}}$, let the relation $\tilde{\mathfrak{y}} \trianglerighteq \mathfrak{y}$ be defined by (7.13). Then the vector space $\boldsymbol{I}_{\mathfrak{l}}^{\mathcal{P}}$ with base $\mathfrak{Y}_{\mathfrak{l},\mathcal{P}}$ is functorial in $\mathcal{P}$ by formula (7.14), and there is a functor isomorphism

$$\boldsymbol{M}^{*\mathcal{P}}_{(n)} \cong \bigoplus_{\mathfrak{l}\in\mathfrak{L}_n(\mathbb{Q})} \boldsymbol{I}_{\mathfrak{l}}^{\mathcal{P}}[-\deg \mathfrak{l}] \tag{7.32}$$

which maps $\mathfrak{y}$ to

$$\bigwedge_{k\in \mathrm{Odd}_{\leq n}} \bigwedge_{l=1}^{\mathfrak{l}(k)} \tilde{\lambda}_k^{\mathfrak{y}(k,l)}.$$

The degree $\deg \mathfrak{l}$ is defined in the same way as for imaginary quadratic fields, by (7.15).

Let

$$\mathfrak{S}_{\mathfrak{l}} = \{(k,l)\,|k \in \mathrm{Odd}_{\leq n}, 1 \leq l \leq \mathfrak{l}(k)\},$$

and let $F(k,l) = k$. As in the case of imaginary quadratic fields, $\boldsymbol{I}_{\mathfrak{l}}^{\bullet}$ can be identified with the antisymmetrization of $\boldsymbol{J}_{\tilde{P}_{n,\mathfrak{S}_{\mathfrak{l}},F,p_{\mathfrak{l}}}}$ with respect to the product of symmetric groups $\prod_{k\in\mathrm{Odd}_{\leq n}} S_{\mathfrak{l}(k)}$. As a result, we get a description for the first summand in (7.30) which is similar to (7.18).

Theorem 7.10. *For $\mathfrak{l} \in \mathfrak{L}_n(\mathbb{Q})$, $\mathfrak{e} \in \{0,1\}$ and $N \leq 0$, let $\mathfrak{X}_{N,\mathfrak{e},\mathfrak{l}}$ be the set of ordered $(N+1)$-tuples $\mathfrak{x} = (X_0,\dots,X_N)$ of subsets of $\{\mathrm{Odd}_{\leq n}\}$ with the following properties:*

- *Each number $k \in \mathrm{Odd}_{\leq n}$ belongs to precisely $\mathfrak{l}(k)$ of the sets X_i;*
- *We have*

$$\sum_{i=0}^{N} \max \#\{X_i\} = n - \mathfrak{e}.$$

If $\mathfrak{l} = 0$, we put $\mathfrak{X}_{N,\mathfrak{e},\mathfrak{l}} = \emptyset$. Then for each $\mathfrak{x} \in \mathfrak{X}_{N,\mathfrak{e},\mathfrak{l}}$, $H^*(C^*(\boldsymbol{I}_{\mathfrak{l}}^{\bullet}))$ has a generator $\{\mathfrak{x}\}$ in degree $N + \mathfrak{e}$, and we have

$$H^i\Big(C^*(\boldsymbol{I}_{\mathfrak{l}}^{\bullet})\Big) = \bigoplus_{\mathfrak{e}=0}^{1} \bigoplus_{\mathfrak{x}\in\mathfrak{X}_{i-\mathfrak{e},\mathfrak{e},\mathfrak{l}}} \mathbb{C}\cdot\{\mathfrak{x}\}.$$

Consequently, the cohomology of the first summand in (7.30) is given by

$$H^j\Big(C^*(\boldsymbol{M}^{*\mathcal{P}}_{(n)})\Big) \cong \bigoplus_{\mathfrak{l}\in\mathfrak{L}_n(\mathbb{K})} \bigoplus_{\mathfrak{e}=0}^{1} \bigoplus_{\mathfrak{x}\in\mathfrak{X}_{j-\mathfrak{e}-\deg \mathfrak{l}}} \mathbb{C}\cdot\{\mathfrak{x}\}.$$

These cohomology classes are given by formulas similar to (7.19) and (7.20), with $\lambda_j^{(i)}$ replaced by $\tilde{\lambda}_j^{(i)}$.

If n is odd, this is the only summand in (7.30), and the computation of $H_c^*(\mathcal{G},\mathbb{C})_{\mathcal{I}}^{\boldsymbol{K}_f}$ is complete in this case. For example, if $n=3$, the only possible $\mathfrak{l}$ is $\mathfrak{l}(3)=1$ or $\mathfrak{l}(3)=0$. In the second case, the $\mathfrak{X}_{N,\mathfrak{e},\mathfrak{l}}$ are empty by definition. In the first case, the only element of the sets $\mathfrak{X}_{N,\mathfrak{e},\mathfrak{l}}$ is $\{\{2\}\}\in\mathfrak{X}_{0,0,\mathfrak{l}}$ which gives us the volume form in degree 5. This compares well to the result of Soulé [21, Theorem 4] which implies that $H^*(\mathrm{SL}_3(\mathbb{Z}),\mathbb{C})$ vanishes in positive dimension. In particular, there are no harmonic cusp forms for $\mathrm{SL}_3(\mathbb{Z})$.

If n is even, then we still have to compute the cohomology of the second summand ${}^e\boldsymbol{M}^{*\mathcal{P}}_{(n)}$ in (7.30). Let ${}^e\mathfrak{L}_n(\mathbb{Q})$ be the set of functions

$$\mathfrak{l}\colon \mathrm{Odd}_{\leq n-1}\to\{0,1,\ldots\}$$

satisfying the condition

$$\sum_{j=1}^{\infty}\Bigl(1+\max\{k\in\mathrm{Odd}_{\leq n-1}\,|\,\mathfrak{l}(k)\geq j\}\Bigr)\leq n. \tag{7.33}$$

If the parabolic subgroup $\mathcal{P}$ corresponds to $0<i_1<\cdots<i_K=n$, let ${}^e\mathfrak{Y}_{\mathfrak{l},\mathcal{P}}$ be empty of one of the numbers i_l is odd, and be equal to the set of functions

$$\mathfrak{y}\colon\ \{(k,l)\,|\,k\in\mathrm{Odd}_{\leq n-1},1\leq l\leq\mathfrak{l}(k)\}\to\{1,\ldots,k\}$$

with the properties (7.11) and (7.12) if all numbers i_k are even. The vector space ${}^e\boldsymbol{I}^{\mathcal{P}}_{\mathfrak{l}}$ with base ${}^e\mathfrak{Y}_{\mathfrak{l},\mathcal{P}}$ is functorial in $\mathcal{P}$ by by formula (7.14), where the relation $\trianglerighteq$ is defined by (7.13), and there is a functor isomorphism

$${}^e\boldsymbol{M}^{*\mathcal{P}}_{(n)}\cong\bigoplus_{\mathfrak{l}\in{}^e\mathfrak{L}_n(\mathbb{Q})}{}^e\boldsymbol{I}^{\mathcal{P}}_{\mathfrak{l}}[-n-\deg\mathfrak{l}] \tag{7.34}$$

which maps $\mathfrak{y}$ to

$$\bigwedge_{j=1}^{K}\varepsilon^{(j)}\wedge\bigwedge_{l=1}^{\mathfrak{l}(k)}\tilde{\lambda}_k^{\mathfrak{y}(k,l)}.$$

Let

$${}^e\mathfrak{S}_{\mathfrak{l}}=\{(k,l)\,|\,k\in\mathrm{Odd}_{\leq n-1},1\leq l\leq\mathfrak{l}(k)\},$$

and let $F(k,l)=k$. Recall the poset $\tilde{P}_{n,{}^e\mathfrak{S}_{\mathfrak{l}},F}$ consisting of partitions of n which have for each $\mathfrak{s}\in{}^e\mathfrak{S}_{\mathfrak{l}}$ a piece of length $\geq F(\mathfrak{s})$ marked, and recall the projection

$$p_{\mathfrak{l}}\colon\tilde{P}_{n,{}^e\mathfrak{S}_{\mathfrak{l}},F}\to\mathfrak{P}$$

which sends a partition of n to the corresponding parabolic subgroup of GL_n. Let $\hat{P}_{n,\mathfrak{l}}\subset\tilde{P}_{n,{}^e\mathfrak{S}_{\mathfrak{l}},F}$ be the subposet of all partitions of n into even pieces, together with a map which for each $\mathfrak{s}\in{}^e\mathfrak{S}_{\mathfrak{l}}$ marks a piece of length $\geq F(\mathfrak{s})$, and let $\hat{p}_{\mathfrak{l}}$ be the restriction of $p_{\mathfrak{l}}$ to $\hat{P}_{n,\mathfrak{l}}$. Then ${}^e\boldsymbol{I}^{\bullet}_{\mathfrak{l}}$ can be identified with the antisymmetrization of $\boldsymbol{J}^{\bullet}_{\hat{P}_{n,\mathfrak{l}},\hat{p}_{\mathfrak{l}}}$ with respect to the product of symmetric groups $\prod_{k\in\mathrm{Odd}_{\le n-1}} S_{\mathfrak{l}(k)}$. Proposition 7.7 can be applied to $\boldsymbol{J}^\bullet_{\tilde{P}_{n,{}^e\mathfrak{S}_\mathfrak{l},F},p_\mathfrak{l}}$ and gives us an isomorphism

$$H^*\left(C^*(\boldsymbol{J}^\bullet_{\hat{P}_{n,\mathfrak{l}},\hat{p}_\mathfrak{l}})\right) \cong \tilde{H}^*(\boldsymbol{B}\hat{P}_{n,\mathfrak{l}}).$$

On the other side, the homotopy type of the poset

$$\hat{P}_{n,\mathfrak{l}} \cong \tilde{P}_{\frac{n}{2},{}^e\mathfrak{S}_\mathfrak{l},\frac{1+F}{2}}$$

is given by Proposition 7.6. We arrive at the following explicit description of the second summand in (7.30).

Theorem 7.11. *If $n = 2$, we have*

$$H^*\left(C^*\left({}^e\boldsymbol{M}^{*\bullet}_{(n)}\right)\right) \cong \mathbb{C}[2].$$

For $n > 2$ and $\mathfrak{l} \in {}^e\mathfrak{L}_n(\mathbb{Q})$, $\mathfrak{e} \in \{0,1\}$, and $N \le 0$, let ${}^e\mathfrak{X}_{N,\mathfrak{e},\mathfrak{l}}$ be the set of ordered $(N+1)$-tuples $\mathfrak{x} = (X_0,\dots,X_N)$ of subsets of $\{\mathrm{Odd}_{\le n-1}\}$ with the following properties:

- *Each number $k \in \mathrm{Odd}_{\le n-1}$ belongs to precisely $\mathfrak{l}(k)$ of the sets X_i;*
- *We have*
$$\sum_{i=0}^{N}(1+\max\{X_i\}) = n - 2\mathfrak{e}.$$

If $\mathfrak{l} = 0$, we put ${}^e\mathfrak{X}_{N,\mathfrak{e},\mathfrak{l}} = \emptyset$. Then for each $\mathfrak{x} \in {}^e\mathfrak{X}_{N,\mathfrak{e},\mathfrak{l}}$, $H^*(C^*({}^e\boldsymbol{I}^\bullet_\mathfrak{l}))$ has a generator $\{\mathfrak{x}\}$ in degree $N+\mathfrak{e}$, and we have

$$H^i\left(C^*\left({}^e\boldsymbol{I}^\bullet_\mathfrak{l}\right)\right) = \bigoplus_{\mathfrak{e}=0}^{1}\ \bigoplus_{\mathfrak{x}\in{}^e\mathfrak{X}_{i-\mathfrak{e},\mathfrak{e},\mathfrak{l}}} \mathbb{C}\cdot\{\mathfrak{x}\}.$$

Consequently, the cohomology of the second summand in (7.30) is given by

$$H^j\left(C^*\left({}^e\boldsymbol{M}^{*\bullet}_{(n)}\right)\right) \cong \bigoplus_{\mathfrak{l}\in{}^e\mathfrak{L}_n(\mathbb{K})}\ \bigoplus_{\mathfrak{e}=0}^{1}\ \bigoplus_{\mathfrak{x}\in{}^e\mathfrak{X}_{j-\mathfrak{e}-n-\deg\mathfrak{l}}} \mathbb{C}\cdot\{\mathfrak{x}\}.$$

Moreover, let the ordering $\prec$ on the roots which was used to define the complex $C^*(\boldsymbol{F}^\bullet)$ be

$$x_1 - x_2 \prec x_2 - x_3 \prec \cdots \prec x_{n-1} - x_n.$$

Then for $\mathfrak{x} = (X_0,\dots,X_N) \in {}^e\mathfrak{X}_{N,0,\mathfrak{l}}$, a representative of the cohomology class $\{\mathfrak{x}\}$ is given by the element

$$\bigwedge_{i=0}^{N}\left(\varepsilon^{(i)} \wedge \bigwedge_{\substack{j=2\\ j\in X_i}}^{\#(X_i)} \tilde{\lambda}_j^{(i)}\right) \tag{7.35}$$

in the cohomology of $\boldsymbol{X}^{(c)}_{\mathcal{M}_{\mathcal{P}}}$*, where* $\mathcal{P} \in \mathfrak{P}$ *is the stabilizer of the standard flag of vector spaces with dimensions*

$$0 < 1 + \#(X_0) < 2 + \#(X_0) + \#(X_1) < \cdots < N - 1 + \sum_{i=0}^{N-2} \#(X_i) < N + \sum_{i=0}^{N-1} \#(X_i) = n.$$

If $\mathfrak{x} = (X_0, \ldots, X_N) \in {}^e\mathfrak{X}_{N,1,\mathfrak{l}}$ *and if* $0 \le k \le N+1$*, then a representative of the cohomology class* $\{\mathfrak{x}\}$ *is given by the element*

$$(-1)^k \bigwedge_{i=0}^{N-1} \varepsilon^{(i)} \bigwedge_{i=0}^{k-1} \bigwedge_{\substack{j=2\\ j \in X_i}}^{\#(X_i)} \tilde{\lambda}_j^{(i)} \wedge \bigwedge_{i=k}^{N} \bigwedge_{\substack{j=2\\ j \in X_i}}^{\#(X_i)} \tilde{\lambda}_j^{(i+1)} \tag{7.36}$$

in the cohomology of $\boldsymbol{X}^{(c)}_{\mathcal{M}_{\mathcal{P}}}$*, where* $\mathcal{P} \in \mathfrak{P}$ *is the stabilizer of the standard flag of vector spaces with dimensions*

$$0 < 1 + \#(X_0) < 2 + \#(X_0) + \#(X_1) < \cdots < k + \sum_{i=0}^{k-1} \#(X_i) < k + 2 + \sum_{i=0}^{k-1} \#(X_i) < \cdots N + 2 + \sum_{i=1}^{N-2} \#(X_i) < 1 + \sum_{i=1}^{n-1} \#(X_i) = n.$$

In the case $n = 4$, we have the vector degree 6 in the first summand in (7.30) defined by $\mathfrak{l}(3) = 1$ and $\mathfrak{x} = \bigl\{\{3\}\bigr\} \in \mathfrak{X}_{0,1,\mathfrak{l}}$. In the second summand, we have the cohomology class defined by $\mathfrak{l}(3) = 1$ and $\mathfrak{x} = \bigl\{\{3\}\bigr\} \in {}^e\mathfrak{X}_{0,0,\mathfrak{l}}$. It is the volume form in degree 9. These are all spherical vectors in the cohomology with compact support, since $H^i\bigl(\mathrm{SL}_4(\mathbb{Z}), \mathbb{Z}\bigr)$ is of dimension one if $i \in \{0; 3\}$ and zero otherwise, by the computation of Lee and Szczarba [16, Theorem 2]. Once again there are no harmonic cusp forms modulo $\mathrm{SL}_4(\mathbb{Z})$. One may ask if this is true for all the groups $\mathrm{SL}_n(\mathbb{Z})$.

It is also possible to give a full computation of $H^*(\mathcal{G}, \mathbb{C})$ for $\mathcal{G} = \mathrm{SL}_n$. The result has a decomposition similar to (7.30) into a summand containing no Euler classes and, for n even, a summand containing the Euler classes. The first of these summands is given by (7.28). The second summand is similar to (7.28), however, the definition of the summands in (7.28) has to be modified to allow only parabolic subgroups corresponding to decompositions of n into even pieces. It is also possible to generalize this to SL_n over arbitrary number fields. The only difference to the cases treated here is that the cohomology with compact support of the Levi components has additional generators in dimension one, which complicate the formulation of the result even more.

Selective Index of Notation

This is a selective index of the mathematical notations which are most frequently used. They are listed according to the order in which they are introduced in the text.

$\boldsymbol{F}(\mathcal{G})^{\bullet}$	section 5, p. 51
$\boldsymbol{G}(\mathcal{G})^{\bullet}_{\bullet}$	section 5, p. 52
$p_{\mathcal{Q}}$	section 6, (6.3), p. 53
$\boldsymbol{H}(\mathcal{G})_{\mathcal{Q}}{}^{*}$	section 6, (6.4), (6.5), pp. 53–54
$\tau_{\mathcal{P}(\mathbb{A}_f)}^{\mathcal{Q}(\mathbb{A}_f)}$	section 6, (6.8), p. 55
$i_{\tilde{\mathcal{Q}}\supseteq\mathcal{Q}}$	section 6, (6.9), p. 55
$P_{n,\mathfrak{S},F}$	subsection 7.3, p. 68
$\tilde{P}_{n,\mathfrak{S},F}$, $M_{n,\mathfrak{S},F,\mathfrak{e},k}$	subsection 7.3, p. 68

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The Saito–Kurokawa Space of $PGSp_4$ and Its Transfer to Inner Forms

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Summary. We discuss the construction, characterization and classification of the Saito–Kurokawa representations of $PGSp_4$ and its inner forms, interpreting them in the framework of Arthur's conjectures. These Saito–Kurokawa representations are among the first examples of the so-called CAP representations or shadows of Eisenstein series.

Introduction

In this paper, we discuss some results on the Saito–Kurokawa space of $PGSp_4$ and its inner forms, interpreting them in the framework of Arthur's conjecture on square-integrable automorphic forms. Most results discussed here are known to the experts—Arthur, Waldspurger, Piatetski-Shapiro, Gelbart, Rallis, Kudla, Moeglin, Soudry—though they may not have been explicitly stated in the literature or written up with the same point of view. The only new material concerns the transfer of the Saito–Kurokawa space to the inner forms and a characterization of the image by means of the standard L-function.

The Saito–Kurokawa cusp forms for $PGSp_4 \cong SO_5$ are the first examples of the so-called CAP representations or shadows of Eisenstein series. They can be constructed (and exhausted) by using theta lifting from $\widetilde{SL}_2$ to SO_5. In the following, we shall first review relevant results for cusp forms on $\widetilde{SL}_2$, and some general results from theta correspondence. We then give a brief description of Arthur's conjecture for $PGSp_4$: this is the natural framework in which the Saito–Kurokawa space can be understood. After describing Piatetski-Shapiro's construction and characterization of the Saito–Kurokawa space for $PGSp_4$, we describe the lifting of the Saito–Kurokawa space to any inner form G' of $PGSp_4$. This is motivated by the recent results of E. Sayag [19]. We end by giving a proof of the characterization of the Saito–Kurokawa space, which is valid for all forms of $PGSp_4$.

Throughout these notes, F will denote a number field and F_v the local field corresponding to a place v of F. The adele ring of F is denoted by $\mathbb{A}$.

1 Waldspurger's results for $\widetilde{SL}_2$

In this section, we review the results of Waldspurger in [24] and [25]. His local results give a partition of the irreducible admissible representations of $\widetilde{SL}_2(F_v)$ into packets. His global results give a complete description of the (genuine) discrete spectrum $L^2_{\text{disc}}(SL_2(F)\backslash\widetilde{SL}_2(\mathbb{A}))$.

1.1 The Weil representations of $\widetilde{SL}_2(F_v)$

We first describe some very special representations of the metaplectic group $\widetilde{SL}_2(F_v)$.

Fix a non-trivial unitary character ψ_v of F_v. Then associated to a quadratic character χ_v of $F_v^\times$ (possibly trivial) is a Weil representation ω_{χ_v} of $\widetilde{SL}_2(F_v)$. The representation can be realized on the space $S(F_v)$ of Schwarz functions on F_v. As a representation of $\widetilde{SL}_2(F_v)$, ω_{χ_v} is reducible. In fact, it is the direct sum of two irreducible representations:

$$\omega_{\chi_v} = \omega^+_{\chi_v} \oplus \omega^-_{\chi_v},$$

where $\omega^+_{\chi_v}$ (resp. $\omega^-_{\chi_v}$) consists of the even (resp. odd) functions in $S(F_v)$. If v is a finite place, then ω^-_χ is supercuspidal and ω^+_χ is not.

1.2 Waldspurger's packets for $\widetilde{SL}_2(F_v)$

In [24] and [25], Waldspurger defined a *surjective* map Wd_{ψ_v} from the set of irreducible genuine (unitary) representations of $\widetilde{SL}_2(F_v)$ which are not equal to $\omega^+_{\chi_v}$ for any χ_v to the set of infinite dimensional (unitary) representations of $PGL_2(F_v)$. We will not go into the definition of Wd_{ψ_v} here. Suffice to say that it involves the study of the local theta correspondence between $\widetilde{SL}_2(F_v)$ and $SO_3(F_v) \cong PGL_2(F_v)$ (cf. Section 2).

In any case, the map Wd_{ψ_v} leads to the following theorem:

Theorem 1.1. *There is a partition of the set of irreducible (unitary) representations of $\widetilde{SL}_2(F_v)$ which are not equal to $\omega^+_{\chi_v}$ for any χ_v, indexed by the infinite dimensional irreducible (unitary) representations of $PGL_2(F_v)$. Namely, if τ_v is such a representation of $PGL_2(F_v)$, we set*

$$\tilde{A}_{\tau_v} = \text{ inverse image of } \tau_v \text{ under } Wd_{\psi_v}.$$

In fact,

$$\#\tilde{A}_{\tau_v} = \begin{cases} 2, & \text{if } \tau_v \text{ is a discrete series;} \\ 1, & \text{if } \tau_v \text{ is not a discrete series.} \end{cases}$$

Moreover, if τ_v is unitary, so are the elements of $\tilde{A}_{\tau_v}$.

In the first case, the set $\tilde{A}_{\tau_v}$ has a distinguished element $\sigma^+_{\tau_v}$, which is characterized by the fact that $\sigma^+_{\tau_v} \otimes \tau_v$ is a quotient of the Weil representation of $\widetilde{SL}_2(F_v) \times SO(2,1)(F_v)$. The other element of $\tilde{A}_{\tau_v}$ will be denoted by $\sigma^-_{\tau_v}$: it is characterized by the fact that $\sigma^-_{\tau_v} \otimes \tau_v$ is a quotient of the Weil representation of $\widetilde{SL}_2(F_v) \times SO(3)(F_v)$ (anisotropic $SO(3)$ here). In the second case, we shall let $\sigma^+_{\tau_v}$ be the unique element in $\tilde{A}_{\tau_v}$ and set $\sigma^-_{\tau_v} = 0$.

Note that this parametrization of the packets on $\widetilde{SL}_2$ in terms of representations of PGL_2 depends on the choice of the character ψ_v. Also, it is quite explicit. For example, for the case $F_v = \mathbb{R}$ and $\psi_v(x) = \exp(2\pi i x)$, if τ_v is the discrete series representation of extremal weights $\pm 2k$ (with k an integer), then $\sigma^+_{\tau_v}$ (resp. $\sigma^-_{\tau_v}$) is the holomorphic (resp. anti-holomorphic) discrete series representation with lowest (resp. highest) weight $k + \frac{1}{2}$.

We also remark that the above discussion in not entirely accurate when $F_v = \mathbb{C}$, for in this case the map Wd_ψ may send some unitary representations of $\widetilde{SL}_2(\mathbb{C}) = SL_2(\mathbb{C}) \times \{\pm 1\}$ to non-unitary representations of $PGL_2(\mathbb{C})$. However, these representations of $\widetilde{SL}_2(\mathbb{C})$ do not intervene in the space of cusp forms and so can be safely ignored for global purposes.

1.3 Cusp forms of $\widetilde{SL}_2(\mathbb{A})$

Let $\widetilde{SL}_2(\mathbb{A})$ be the two-fold cover of $SL_2(\mathbb{A})$, and fix a non-trivial unitary character $\psi = \prod_v \psi_v$ of $F \backslash \mathbb{A}$. Let $\widetilde{\mathcal{A}}_2$ denote the space of square-integrable genuine automorphic forms on $\widetilde{SL}_2(\mathbb{A})$. Then there is an orthogonal decomposition

$$\widetilde{\mathcal{A}}_2 = \widetilde{\mathcal{A}}_{00} \bigoplus \left(\bigoplus_\chi \widetilde{\mathcal{A}}_\chi \right).$$

Here, χ runs over all quadratic characters (possibly trivial) of $F^\times \backslash \mathbb{A}^\times$.

Now the space $\bigoplus_\chi \widetilde{\mathcal{A}}_\chi$ is what people call the space of "elementary theta functions." It is a space which is very well-understood. Indeed, let us describe the space $\widetilde{\mathcal{A}}_\chi$ more concretely. If $\omega_\chi = \otimes_v \omega_{\chi_v}$ is the global Weil representation attached to χ, then the formation of the theta series gives a map

$$\theta_\chi : \omega_\chi \to \widetilde{\mathcal{A}}_2,$$

whose image is the space $\widetilde{\mathcal{A}}_\chi$. To describe the decomposition of $\widetilde{\mathcal{A}}_\chi$, for a finite set S of places of F, let us set

$$\omega_{\chi,S} = (\otimes_{v \in S} \omega^-_{\chi_v}) \otimes (\otimes_{v \notin S} \omega^+_{\chi_v})$$

so that

$$\omega_\chi = \bigoplus_S \omega_{\chi,S}.$$

Then we have

$$\widetilde{\mathcal{A}}_\chi \cong \bigoplus_{\#S \text{ even}} \omega_{\chi,S}.$$

Moreover, $\omega_{\chi,S}$ is cuspidal if and only if S is non-empty.

1.4 Multiplicity-one result

Thus, the main problem in the study of cusp forms on $\widetilde{SL}_2(\mathbb{A})$ is the description of $\widetilde{\mathcal{A}}_{00}$. In [25], Waldspurger showed:

Theorem 1.2. $\widetilde{\mathcal{A}}_{00}$ *(and also* $\widetilde{A}_2$*) satisfies multiplicity one.*

This theorem is proved by studying the global theta correspondence for $\widetilde{SL}_2$ and $SO_3 \cong PGL_2$, and then appealing to the multiplicity-one theorem for PGL_2.

1.5 Near equivalence classes

Note that $\widetilde{SL}_2$ does not satisfy strong multiplicity one: there are non-isomorphic cuspidal representations π_1 and π_2 whose local components $\pi_{1,v}$ and $\pi_{2,v}$ are isomorphic for almost all places v. We say that such π_1 and π_2 are *nearly equivalent*; this is an equivalence relation on abstract representations. In the paper [24], Waldspurger described the near equivalence classes of representations in $\widetilde{\mathcal{A}}_{00}$. Let us describe his results.

Given a cuspidal automorphic representation $\tau = \otimes_v \tau_v$ of PGL_2, we define a set of irreducible unitary representations of $\widetilde{SL}_2(\mathbb{A})$ as follows. Recall that for each place v, we have a local "packet"

$$\tilde{A}_{\tau_v} = \{\sigma^+_{\tau_v}, \sigma^-_{\tau_v}\}$$

where $\sigma^-_{\tau_v} = 0$ if τ_v is not a discrete series. Now set

$$\tilde{A}_\tau = \{\sigma = \otimes_v \sigma^{\epsilon_v}_{\tau_v} : \epsilon_v = \pm \text{ and } \epsilon_v = + \text{ for almost all } v\}.$$

This is the global "packet" of $\widetilde{SL}_2(\mathbb{A})$ associated to the cuspidal representation τ of PGL_2.

For

$$\sigma = \otimes_v \sigma^{\epsilon_v}_{\tau_v} \in \tilde{A}_\tau,$$

let us set

$$\epsilon_\sigma = \prod_v \epsilon_v.$$

Then we have [24, p. 286, Corollaries 1 and 2]:

Theorem 1.3.

$$\widetilde{\mathcal{A}}_{00} = \bigoplus_{\text{cuspidal},\tau} \widetilde{\mathcal{A}}(\tau)$$

where each $\widetilde{\mathcal{A}}(\tau)$ *is a near equivalence class of cuspidal representations and is given by:*

$$\widetilde{\mathcal{A}}(\tau) = \bigoplus_{\sigma\in\tilde{A}_\tau:\epsilon_\sigma=\epsilon(\tau,1/2)} \sigma.$$

1.6 Remarks

Note that in the above theorem, there are some τ which have $\widetilde{\mathcal{A}}_\tau = 0$. Indeed, this happens precisely when τ_v is the principal series for all v (so that the global packet $\tilde{A}_\tau$ is a singleton set) and $\epsilon(\tau, 1/2) = -1$. Such τ can also be characterized by the fact that (cf. [24, Lemma 41, p. 282])

$$\epsilon(\tau\otimes\chi, 1/2) = -1 \qquad \text{for any quadratic character } \chi \text{ of } F^\times\backslash\mathbb{A}^\times.$$

2 Theta correspondence

In this section, we give a brief discussion of the basic setup and results of classical theta correspondence. For the general idea, the articles in Corvallis suffice. For a more detailed treatment of the local theory, one can consult the notes of Kudla [6] or the article by Li [12].

2.1 Dual pairs

Suppose that $Sp(\mathbb{W})$ is a symplectic group. We are interested in reductive subgroups G_1 and G_2 of $Sp(\mathbb{W})$ such that each is the centralizer of the other. Such a pair of subgroups is called a dual pair. The possible dual pairs in $Sp(\mathbb{W})$ have been classified by Howe. The standard example is obtained as follows. If (V, q) is a quadratic space and W a symplectic space, then $\mathbb{W} := V \otimes W$ becomes a symplectic space naturally, and we have natural inclusions

$$O(V, q) \hookrightarrow Sp(\mathbb{W}) \quad \text{and} \quad Sp(W) \hookrightarrow Sp(\mathbb{W}).$$

The groups $O(V, q)$ and $Sp(W)$ form a dual pair.

2.2 Double covers

Now assume we are working over a local field F_v. The symplectic group $Sp(\mathbb{W}_v)$ has a unique (non-linear) double cover $\widetilde{Sp}(\mathbb{W}_v)$ (unless $F_v = \mathbb{C}$).

This double cover always splits over the subgroup $O(V,q)$. The splitting may not be unique, but using the quadratic form q, one can specify a particular splitting. The cover splits over $Sp(W)$ if and only if $\dim(V)$ is even. In the case of interest in these notes, $\dim(V) = 5$ and $\dim(W) = 2$. Thus, we are forced to work with non-linear groups. In particular, we have:

$$G_1 \times G_2 := \widetilde{Sp}(W) \times SO(V,q) \longrightarrow \widetilde{Sp}(\mathbb{W}_v).$$

2.3 Weil representations

Fix a non-trivial additive character ψ of F_v. Then $\widetilde{Sp}(\mathbb{W}_v)$ has a representation ω_{ψ_v} called the Weil representation associated to ψ_v. Like the case of $\widetilde{SL}_2(F_v)$ discussed earlier, it is the sum of two irreducible representations. We may pullback this representation to the group $G_{1,v} \times G_{2,v}$. The resulting representation is denoted by $\omega_{\psi_v,q}$ (since it depends on the choice of splitting over $SO(V,q)$). The representation $\omega_{\psi,q}$ have various concrete realizations which are essential in applications. For the cases of interest here, one can find formulas for these realizations of $\omega_{\psi,q}$ in [3, §3] among other places.

2.4 Local Howe conjecture

Now $\omega_{\psi,q}$ is of course highly reducible and we are interested in how it breaks up into irreducibles. Since we are not working with finite groups here, we need to formulate more precisely what we mean. Suppose that π_v is an irreducible representation of $G_{1,v}$. We let $\omega_{\psi_v,q}[\pi_v]$ denote the maximal π_v-isotypic quotient of $\omega_{\psi_v,q}$. Thus

$$\omega_{\psi_v,q}[\pi_v] = \omega_{\psi_v,q} / \bigcap_{\phi} \mathrm{Ker}(\phi)$$

where ϕ runs over all non-zero equivariant maps $\phi : \omega_{\psi_v,q} \to \pi_v$. As a representation of $G_{1,v}$, it is an isotypic sum of π_v.

Now because $G_{1,v}$ commutes with $G_{2,v}$, $\omega_{\psi_v,q}[\pi_v]$ also inherits an action of $G_{2,v}$, and as a representation of $G_{1,v} \times G_{2,v}$, can be defined as:

$$\omega_{\psi_v,q}[\pi_v] = \pi_v \otimes \theta(\pi_v)$$

for some smooth (possibly zero) representation $\theta(\pi_v)$ of $G_{2,v} = SO(V,q)$. We further let $\Theta(\pi_v)$ be the maximal semisimple quotient of $\theta(\pi_v)$. Clearly, we have analogous definitions with the roles of G_1 and G_2 reversed.

The following is a conjecture of Howe:

Howe's Conjecture for the pair $G_1 \times G_2$: For any representation π_v of G_1, either $\theta(\pi_v)$ is zero or else its maximal semisimple quotient $\Theta(\pi_v)$ is non-zero irreducible. Moreover, the analogous statement with the roles of G_1 and G_2 exchanged also holds.

A trivial reformulation of this conjecture is: for each representation π_v of $G_{1,v}$, there exists at most one irreducible representation τ_v of $G_{2,v}$ such that

$$\mathrm{Hom}_{G_{1,v}\times G_{2,v}}(\omega_{\psi_v,q},\pi_v\otimes\tau_v)\neq 0.$$

If such a τ_v exists (call it $\Theta(\pi_v)$), the above Hom space is one-dimensional. Moreover, if $\Theta(\pi_v)\cong\Theta(\pi'_v)$ are non-zero, then $\pi_v\cong\pi'_v$.

In fact, this conjecture is almost totally proved:

Theorem 2.1. *(i) Howe's conjecture is true over all archimedean local fields (as proved by Howe) and all p-adic fields with $p\neq 2$ (as proved by Waldspurger [23]).*

(ii) For any p, Howe's conjecture for π_v is true if π_v is supercuspidal (as proved by Kudla [7]); i.e., either $\Theta(\pi_v)=0$ or $\Theta(\pi_v)$ is irreducible.

There are of course other isolated cases (of pairs $G_1\times G_2$) for which the conjecture is proven; some instances may be found in [16]. In the case of interest in $\widetilde{SL}_2\times SO(5)$, this remaining case of $p=2$ can be checked directly by hand. Thus in the rest of these notes, we shall assume that the local Howe conjecture is known. A corollary is:

Corollary 2.2. *The map $\pi_v\mapsto\Theta(\pi_v)$ gives an* injective *map Θ_v from a subset of the admissible dual of $\widetilde{Sp}(W)$ to the admissible dual of $SO(V,q)$ (namely, Θ_v is defined on those π_v such that $\Theta(\pi_v)\neq 0$). This map is called the* local theta lift.

2.5 Remarks

Note that the local theta correspondence depends on the choices of the character ψ_v and the quadratic form q (and not just on the orthogonal group). However, $\omega_{\psi_\lambda,q}=\omega_{\psi,\lambda q}$. The main unsolved problem in local theta correspondence is the explicit description of $\Theta(\pi_v)$ given π_v.

2.6 Stable range

There are certain favorable circumstances which ensure that $\Theta(\pi_v)$ is non-zero for any π_v, so that the map Θ_v in the above corollary is defined on the whole admissible dual. One such example is the case of the *stable range*.

Definition 2.3. $(Sp(W),SO(V,q))$ *is in the stable range (with $Sp(W)$ the smaller group) if (V,q) contains an isotropic subspace whose dimension is $\geq\dim(W)$.*

For example, if V is split of dimension 5, then $(SL_2,SO(V))$ is in the stable range.

Theorem 2.4. *Suppose that (G_1,G_2) is in the stable range with G_1 the smaller group. Then for any representation π_v of $G_{1,v}$, $\Theta(\pi_v)\neq 0$ (cf. [7]). Moreover, if π_v is unitary, then so is $\Theta(\pi_v)$ (this was proved by Li [10]).*

2.7 Global Theta lift

Now we come to the global setting. Fix a non-trivial additive character ψ of $F\backslash\mathbb{A}$. Then we have the global Weil representation of $\widetilde{Sp}(\mathbb{W})(\mathbb{A})$

$$\omega_\psi = \otimes_v \omega_{\psi_v}.$$

By pulling back to $G_{1,\mathbb{A}} \times G_2(\mathbb{A})$, we have the representation $\omega_{\psi,q}$.

It turns out that there is a natural map

$$\theta : \omega_\psi \longrightarrow \mathcal{A}_2(\widetilde{Sp}(\mathbb{W}))$$

of ω_ψ to the space of square-integrable automorphic forms on $\widetilde{Sp}(\mathbb{W})$. For $\varphi \in \omega_\psi$, we may pullback the function $\theta(\varphi)$ to the group $\widetilde{Sp}(W)(\mathbb{A}) \times SO(V,q)(\mathbb{A})$. This function is of moderate growth on the adelic points of the dual pair. The space of functions thus obtained is a quotient of the representation $\omega_{\psi,q}$.

Now let $\pi \subset \mathcal{A}(G_{1,\mathbb{A}})$ be a cuspidal representation. For $f \in \pi$ and $\varphi \in \omega_{\psi,q}$, we set:

$$\theta(\varphi, f)(g) = \int_{G_{1,F}\backslash G_{1,\mathbb{A}}} \theta(\varphi)(gh) \cdot \overline{f(h)}\, dh.$$

Then $\theta(\varphi, f)$ is an automorphic form on $G_2 = SO(V,q)$. Denote the space of automorphic forms spanned by the $\theta(\varphi, f)$ for all φ and f by $V(\pi)$; it is a $G_2(\mathbb{A})$-submodule in $\mathcal{A}(G_2)$ and is called the *global theta lift* of π.

A main question in theta correspondence is to decide if $V(\pi)$ is non-zero. When (G_1, G_2) is in the stable range, then in fact one can show that $V(\pi)$ is always non-zero.

2.8 Local-global compatibility

How is the representation $V(\pi)$ related to the irreducible representation $\Theta(\pi) := \otimes_v \Theta(\pi_v)$? We have:

Proposition 2.5. *Assume the local Howe conjecture holds for the pair $G_1 \times G_2$ at every place v. Suppose that $V(\pi)$ is non-zero and is contained in the space of square-integrable automorphic forms on G_2. Then $V(\pi) \cong \Theta(\pi)$.*

Proof. We are told that $V(\pi)$ is semisimple. Let τ be an irreducible summand of $V(\pi)$. Then consider the linear map

$$\omega_{\psi,q} \otimes \pi^\vee \otimes \tau^\vee \longrightarrow \mathbb{C}$$

defined by:

$$\varphi \otimes \overline{f_1} \otimes \overline{f_2} \mapsto \int_{G_2(F)\backslash G_2(\mathbb{A})} \theta(\varphi, f_1)(g) \cdot \overline{f_2(g)}\, dg.$$

This map is non-zero and $(G_{1,\mathbb{A}} \times G_2(\mathbb{A}))$-equivariant. Thus it gives rise to a non-zero equivariant map

$$\omega_{\psi,q} \longrightarrow \pi \otimes \sigma,$$

and thus for all v, a non-zero $(G_{1,v} \times G_{2,v})$-equivariant map

$$\omega_{\psi_v,q} \longrightarrow \pi_v \otimes \tau_v.$$

In other words, we must have

$$\tau_v \cong \Theta(\pi_v).$$

Hence, $V(\pi)$ must be an isotypic sum of $\Theta(\pi)$. Moreover, the multiplicity-one statement in Howe's conjecture implies that

$$\dim \mathrm{Hom}_{G_{1,\mathbb{A}} \times G_2(\mathbb{A})}(\omega_{\psi,q}, \pi \otimes \Theta(\pi)) = 1.$$

Thus $V(\pi)$ is in fact irreducible and isomorphic to $\Theta(\pi)$. □

2.9 Multiplicity preservation

Note that because the local theta lifting is an injective map on its domain, we know that if π_1 and π_2 are two cuspidal representations which are non-isomorphic, then $V(\pi_1) \ncong V(\pi_2)$. We come now to the question of multiplicity preservation, first observed by Rallis in [16]. Suppose that the multiplicity of an abstract representation π in the space of cusp forms of G_1 is $m_{cusp}(\pi)$. Let $\mathcal{A}_{cusp}(\pi)$ be the π-isotypic subspace. We have:

Proposition 2.6. *Suppose that:*

- *the local Howe conjecture holds for the pair $G_1 \times G_2$ at every place v;*
- *for any irreducible summand $\pi_0 \subset \mathcal{A}_{cusp}(\pi)$, $V(\pi_0)$ is non-zero and is contained in the space of square-integrable automorphc forms on G_2.*

Then the multiplicity of the irreducible representation $\Theta(\pi)$ in $V(\mathcal{A}_{cusp}(G_1))$ is equal to $m_{cusp}(\pi)$.

Proof. The proof is similar to that of the previous proposition. For simplicity, let us denote

$$V_1 = \mathrm{Hom}_{G_1}(\pi, \mathcal{A}_{cusp}(G_1))$$

and

$$V_2 = \mathrm{Hom}_{G_2}(\Theta(\pi), V(\mathcal{A}_{cusp}(\pi))).$$

Because of the injectivity of the map Θ on its domain,

$$V_2 = \mathrm{Hom}_{G_2}(\Theta(\pi), V(\mathcal{A}_{cusp}(G_1))).$$

We need to show that V_1 and V_2 have the same dimension. Consider the pairing:

$$\langle -, - \rangle : V_1 \times V_2 \longrightarrow \mathrm{Hom}_{G_1 \times G_2}(\omega_{\psi,q} \otimes \pi^\vee \otimes \Theta(\pi)^\vee, \mathbb{C}) \cong \mathbb{C}$$

given by:

$$\langle f_1, f_2\rangle(\varphi \otimes v_1 \otimes v_2) = \int_{G_2(F)\backslash G_2(\mathbb{A})} \theta(\varphi, f_1(v_1))(g) \cdot \overline{f_2(v_2)(g)}\, dg.$$

(Note that $\pi_i^\vee \cong \overline{\pi_i}$ because π_i is unitary). The target space of this pairing is isomorphic to $\mathbb{C}$ because of the local Howe conjecture, which we assume to be true. Now it is easy to see that this pairing is perfect and thus exhibit V_1 and V_2 as linear dual of each other. This proves the proposition. □

2.10 Rallis inner product formula

There is a very beautiful formula for the inner product $\langle \theta(\varphi, f), \theta(\varphi, f)\rangle_{G_2}$ (assuming that it converges absolutely) which was first obtained by Rallis ([15] and [17]) for certain cases. In [11], this was extended to more cases. For the case at hand, namely for the dual pair $\widetilde{SL}_2 \times SO(V, q)$ with $\dim(V)$ odd, the results we need can be found in [17, Theorem 6.2, p. 178].

Theorem 2.7. *Suppose that* $\dim(V) = n \geq 3$ *is odd. Let* σ *be a cuspidal representation of* $\widetilde{SL}_2$ *contained in* $\widetilde{A}_{00}$*, so that* σ *is in a global Waldspurger packet associated to a cuspidal representation* τ *of* PGL_2*. There is a finite set of places, including the archimedean ones and the places dividing* 2*, such that*

$$\langle \theta(\varphi_1, f_1), \theta(\varphi_2, f_2)\rangle = \left(\prod_{v\in S} I_v(\varphi_{1,v} \otimes f_{1,v}, \varphi_{2,v} \otimes f_{2,v})\right) \cdot \frac{L^S(\tau \otimes \chi_{\mathrm{disc}(q)}, \frac{n-2}{2})}{\zeta^S(\frac{n-2}{2})\zeta^S(n-1)}.$$

Here,

$$I_v(\varphi_{1,v} \otimes f_{1,v}, \varphi_{2,v} \otimes f_{2,v}) = \int_{\widetilde{SL}(F_v)} \langle \omega_{\psi,q}(h)\varphi_1, \varphi_2\rangle \cdot \langle f_2, \overline{\sigma_v}(h) f_1\rangle\, dh.$$

is a sesquilinear form on $\omega_{\psi_v,q} \otimes \overline{\sigma_v}$ *and the* $\langle -, -\rangle$ *in the integral on the right refers to inner products on* $\omega_{\psi_v,q}$ *and* σ_v *(conjugate linear in the second argument).*

Note that if $n \geq 5$, the special L-values are all non-vanishing. Thus the non-vanishing of the global theta lift depends on the non-vanishing of the finitely many local sesquilinear forms I_v. This question was addressed in [17, Proposition 6.1 and Corollary 6.1]:

Theorem 2.8. *The form* I_v *is non-zero if and only if* $\Theta_v(\sigma_v) \neq 0$*, i.e. the local theta lift of* σ_v *is non-zero.*

Conclusion 2.9. The point of the two theorems above is that when $\dim(V) \geq 5$, the non-vanishing of the global theta lift depends entirely on the non-vanishing of the local theta lifts at all places (or rather at the finitely many places where σ_v is not unramified). Thus it is purely a local problem. When $\dim(V) = 3$, notice that we will get $L^S(\tau \otimes \chi_{\mathrm{disc}(q)}, 1/2)$ in the inner product formula; thus in this case, there will be a global obstruction to the non-vanishing of global theta lift. This was actually first observed in Waldspurger's work [25].

3 Arthur's conjecture on the discrete spectrum of $PGSp(4)$

In this section, we review what Arthur's conjecture says for the discrete spectrum L^2_{disc} of $G = PGSp_4$. Loosely speaking, Arthur's conjecture is a classification of the near equivalence classes of representations in $L^2_{\mathrm{disc}}(G(F)\backslash G(\mathbb{A}))$, in the spirit of the theorem of Waldspurger for $\widetilde{SL}_2$. We will relate this framework of Arthur to the notion of CAP (cuspidal associated to parabolics) representations (introduced by Piatetski-Shapiro). The references for Arthur's conjecture are of course [1] and [2].

3.1 A basic assumption

In the formulation of Arthur's conjecture, one needs to make a serious assumption:

Assumption 3.1. There is a topological group L_F (depending only on the number field F) satisfying the following properties:

- the identity component L_F^0 of L_F is compact and the group of components $L_F^0\backslash L_F$ is isomrophic to the Weil group W_F;
- for each place v, there is a natural conjugacy class of embeddings $L_{F_v} \hookrightarrow L_F$, where L_{F_v} is the Weil group if F_v is archimedean, and the Weil–Deligne group $W_{F_v} \times SL_2(\mathbb{C})$ if F_v is non-archimedean.
- there is a natural bijection between the set of isomorphism classes of irreducible representations of L_F of dimension n and the set of cuspidal representations of $GL_n(\mathbb{A})$.

This assumption is basically the main conjecture in the Langlands program for GL_n.

3.2 A-parameters

By a (discrete) A-parameter for G, we mean an equivalence class of maps

$$\psi : L_F \times SL_2(\mathbb{C}) \longrightarrow \hat{G} = Sp_4(\mathbb{C})$$

which satisfy some conditions:

- the restriction of ψ to L_F has bounded image (i.e., gives a tempered L-parameter);
- the restriction of ψ to $SL_2(\mathbb{C})$ is an algebraic homomorphism;
- the centralizer of the image of ψ in $Sp_4(\mathbb{C})$ is finite (so the image of ψ is not too small). We let Z_ψ be this centralizer and let S_ψ denote the quotient of Z_ψ by the center $Z_{\hat{G}} = \{\pm 1\}$ of $Sp_4(\mathbb{C})$.

3.3 Decomposition of discrete spectrum

Now according to Arthur, the discrete spectrum possesses a decomposition

$$L^2_{\text{disc}}(G(F)\backslash G(\mathbb{A})) = \widehat{\bigoplus}_\psi L^2_\psi,$$

where the Hilbert space direct sum runs over the A-parameters ψ. For any ψ, the space L^2_ψ will be a direct sum of nearly equivalent representations, and we want to describe its internal structure next.

3.4 Local A-packets

The global A-parameter ψ gives rise to a local A-parameter

$$\psi_v : L_{F_v} \times SL_2(\mathbb{C}) \longrightarrow \hat{G}$$

for each place v of F. Denote by Z_{ψ_v} the centralizer of the image of ψ_v, and set $S_{\psi_v} = \pi_0(Z_{\psi_v}/Z_{\hat{G}}) = Z_{\psi_v}/Z^0_{\psi_v} Z_{\hat{G}}$.

Arthur speculated that to each irreducible representation η_v of S_{ψ_v}, one can attach a unitary admissible (possibly reducible, possibly zero) representation π_{η_v} of $G(F_v)$. Thus we have a finite set

$$A_{\psi_v} = \left\{ \pi_{\eta_v} \, : \, \eta_v \in \widehat{S}_{\psi_v} \right\}.$$

This is the local A-packet attached to ψ_v. Of course, there are some conditions to satisfy:

- for almost all v, π_{η_v} is irreducible and unramified if η_v is the trivial character 1_v. For such v, π_{1_v} is the unramified representation whose Satake parameter is:
$$s_{\psi_v} = \psi_v \left(Fr_v \times \begin{pmatrix} q_v^{1/2} & \\ & q_v^{-1/2} \end{pmatrix} \right),$$
where Fr_v is a Frobenius element at v and q_v is the number of elements of the residue field at v. In fact, it is required that for any v, π_{1_v} has Langlands parameter ϕ_{ψ_v} given by:

$$\phi_{\psi_v}(w) = \psi_v\left(w \times \begin{pmatrix} |w|_v^{1/2} & \\ & |w|_v^{-1/2} \end{pmatrix}\right)$$

for any $w \in L_{F_v}$.

- a number of other conditions concerning the character distributions which are too technical to state and which will not concern us here.

These requirements may not characterize the set A_{ψ_v} but they come pretty close. The main point to notice here is that for almost all v, we know what the representation π_{1_v} is. However, as it stands, the conjecture does not specify what the other representations in the local packets are. For example, we are not told what are their L-parameters. However, I learned the following conjecture from Moeglin:

Conjecture 3.2. *If v is a finite place and*

$$\psi_v : (W_{F_v} \times SL_2(\mathbb{C})) \times SL_2(\mathbb{C}) \longrightarrow \hat{G}$$

is a local A-parameter with associated local A-packet A_{ψ_v}, then any discrete series representation in A_{ψ_v} has L-parameter equal to

$$W_{F_v} \times SL_2(\mathbb{C}) \xrightarrow{id \times \Delta} W_{F_v} \times SL_2(\mathbb{C}) \times SL_2(\mathbb{C}) \xrightarrow{\psi_v} \hat{G}.$$

3.5 Global A-packets

With the local packets A_{ψ_v} at hand, we may define the global A-packet by:

$$A_\psi = \{\pi = \otimes_v \pi_{\eta_v} : \pi_{\eta_v} \in A_{\psi_v}\, \eta_v = 1_v \text{ for almost all } v\}.$$

Observe that this is a set of nearly equivalent representations of $G(\mathbb{A})$, indexed by the irreducible representations of the compact group

$$\mathcal{S}_{\psi,\mathbb{A}} := \prod_v S_{\psi,v}.$$

Note that there is a diagonal map

$$\Delta : S_\psi \longrightarrow S_{\psi,\mathbb{A}}.$$

If $\eta = \bigotimes_v \eta_v$ is an irreducible character of $\mathcal{S}_{\psi,\mathbb{A}}$, then we may set

$$\pi_\eta = \bigotimes_v \pi_{\eta_v}.$$

This is possible because for almost all v, $\eta_v = \mathbf{1_v}$ and $\pi_{\mathbf{1_v}}$ is required to be unramified.

3.6 Multiplicity formula

The space L^2_ψ will be the sum of the elements of A_ψ with some multiplicities. More precisely, Arthur attached to ψ a quadratic character ϵ_ψ of S_ψ; we will not give the definition here, but will say what it is when we discuss the Saito–Kurokawa representations in Section 4. Now if η is an irreducible character of $S_{\psi,\mathbb{A}}$, we set

$$m_\eta = \langle \Delta^*(\eta), \epsilon_\psi \rangle_{S_\psi} = \frac{1}{\# S_\psi} \cdot \left(\sum_{s \in S_\psi} \epsilon_\psi(s) \cdot \eta(s) \right).$$

Then Arthur conjectures that

$$L^2_\psi = \bigoplus_\eta m_\eta \pi_\eta.$$

3.7 Inner forms

We should state that the above description of Arthur's conjecture is only accurate for split groups (though it can be extended to quasi-split groups naturally). For inner forms of a split group, some modifications are necessary; we indicate these briefly.

A global A-parameter ψ for a split group G is also an A-parameter for an inner form G' provided that ψ is relevant, i.e., its image is not contained in the Levi of an irrelevant parabolic subgroup. We saw above that the representations in the local packet for $G(F_v)$ are indexed by irreducible characters η_v of $Z_{\psi_v}/Z^0_{\psi_v} Z_{\hat{G}}$. We should think of η_v as a character of $Z_{\psi_v}/Z^0_{\psi_v}$ which is trivial on $Z_{\hat{G}}$.

Now the main modification for G' is that the representations in the local packet of $G'(F_v)$ should be indexed by (some of) the characters of $Z_{\psi_v}/Z^0_{\psi_v}$ which are *non-trivial* on $Z_{\hat{G}}$, at least not when G is an adjoint group.

The definition of the quadratic character ϵ_ψ does not change; thus ϵ_ψ is a quadratic character of Z_ψ which is trivial on $Z_{\hat{G}}$. For a representation π_η in the global A-packet, where η is an irreducible character of $Z_{\psi,\mathbb{A}}$, the multiplicity $m(\pi_\eta)$ is given by $\langle \epsilon_\psi, \Delta^*(\eta) \rangle_{Z_\psi}$. Note that this is non-zero only if $\Delta^*(\eta)$ is trivial on $Z_{\hat{G}}$. We do not know if this condition is automatic (though for the case of interest in these notes, it is).

3.8 The A-parameters of $PGSp_4$

Now we want to see more concretely what Arthur's conjecture says for $PGSp_4$. We first describe the A-parameters of $PGSp_4$.

We can first partition the set of A-parameters ψ of $PGSp_4$ according to the restriction of ψ to $SL_2(\mathbb{C})$. Recall that the Jacobson–Morozov theorem states that there is a bijection between the set of conjugacy classes of homomorphisms $SL_2(\mathbb{C}) \longrightarrow \hat{G}$ and the set of unipotent conjugacy classes in $\hat{G}$. The bijection is given by attaching to a morphism $\iota : SL_2 \to \hat{G}$ the conjugacy class of the unipotent element

$$\iota \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Now $\hat{G} = Sp_4(\mathbb{C})$ has four unipotent conjugacy classes, and thus there are four families of A-parameters. There is a partial order on the set of unipotent classes:

$$\mathcal{O}_1 < \mathcal{O}_2 \quad \text{if and only if} \quad \mathcal{O}_1 \subset \overline{\mathcal{O}_2}.$$

We arrange the four unipotent classes in increasing sizes:

$$\mathcal{O}_0 = \{1\} < \mathcal{O}_{\text{long}} < \mathcal{O}_{\text{short}} < \mathcal{O}_{\text{reg}}.$$

Here $\mathcal{O}_0$ is the trivial class, $\mathcal{O}_{\text{long}}$ is the class of a non-trivial element in a long-root-subgroup, $\mathcal{O}_{\text{short}}$ is the class of a non-trivial element in a short-root-subgroup and $\mathcal{O}_{\text{reg}}$ is the principal (or regular) unipotent conjugacy class.

One can describe the morphism ι_* attached to the conjugacy class $\mathcal{O}_*$ more concretely. Obviously, ι_0 is the trivial map. Now there is a natural embedding

$$j : SL_2(\mathbb{C}) \times SL_2(\mathbb{C}) \hookrightarrow Sp_4(\mathbb{C})$$

and we have:

$$\begin{cases} \iota_{\text{long}}(g) = j(g, 1); \\ \iota_{\text{short}}(g) = j(g, g). \end{cases}$$

Finally, ι_{reg} gives the action of SL_2 on its irreducible four-dimensional representation.

This partition of the set of A-parameters leads to a decomposition of the discrete spectrum into four pieces:

$$L^2_{\text{disc}} = \mathcal{A}_0 \oplus \mathcal{A}_{\text{long}} \oplus \mathcal{A}_{\text{short}} \oplus \mathcal{A}_{\text{reg}}.$$

We consider these four pieces separately:

- if ψ corresponds to the orbit $\mathcal{O}_{\text{reg}}$, then the local and global A-packets are singletons, consisting of one-dimensional representations (quadratic characters). Thus $\mathcal{A}_{\text{reg}}$ is the direct sum of quadratic Grossencharacters.
- $\mathcal{A}_{\text{long}}$ is the so-called Saito–Kurokawa space (as explained below). This space was constructed by Piatetski-Shapiro [13] and we will review his results below. We shall call the A-parameters in this class the *Saito–Kurokawa parameters.*
- $\mathcal{A}_{\text{short}}$ was constructed by Howe–Piatetski-Shapiro [5] and Soudry [22].

- $\mathcal{A}_0$ is conjecturally the tempered part of $L^2_{\rm disc}$. Thus it is the most nondegenerate part of $L^2_{\rm disc}$. Note that $\mathcal{A}_0$ decomposes naturaly into two parts, depending on whether the tempered parameter ψ factors through the subgroup $SL_2(\mathbb{C}) \times SL_2(\mathbb{C})$. If it does, we say that ψ is tempered endoscopic. If it doesn't, we say that ψ is stable. Thus we have $\mathcal{A}_0 = \mathcal{A}_{0,\rm end} \oplus \mathcal{A}_{0,\rm st}$. The space $\mathcal{A}_{0,\rm end}$ was first studied by Yoshida [26] (for classical Siegel-modular forms) and the general construction has been carried out by Roberts [18].

3.9 CAP representations

Most of the constructions mentioned in the previous subsection were couched in the language of CAP representations, rather than in the framework of Arthur's conjectures.

Definition 3.3. *A cuspidal representation π of a* quasi-split *group G is said to be* CAP *with respect to a parabolic subgroup P of G if it is nearly equivalent to the irreducible constituents of an induced representation $\mathrm{Ind}_P^G \tau$, with τ a cuspidal representation of the Levi factor of P.*

In particular, a CAP representation is nearly equivalent to an Eisenstein series. This explains why CAP representations are sometimes called *shadows of Eisenstein series.*

3.10 Inner forms

The above definition is not the right notion for non-quasi-split groups. Here is the right definition. If G' is an inner form of a quasi-split G, then G'_v and G_v are isomorphic for almost all v. Thus it makes sense to say that a representation π' of $G'(\mathbb{A})$ is nearly equivalent to a representation π of $G(\mathbb{A})$. Thus we say that a cuspidal π' of G' is CAP with respect to a parabolic P of G if π' is nearly equivalent to the constituents of $\mathrm{Ind}_P^G \tau$ with τ cuspidal. That this modification of the notion of CAP is necessary (and reasonable) is suggested by the paper of Sayag [19].

We make another definition:

Definition 3.4. *The Saito–Kurokawa space of $PGSp_4$ (or its inner forms) is the subspace $\mathcal{A}_{SK} \subset L^2_{\rm disc}$ consisting of all representations which are CAP with respect to the Siegel parabolic P.*

We note that Arthur's conjecture *predicts* that the cuspidal part of $\mathcal{A}_{long}$ is precisely the subspace of representations which are CAP with respect to the Siegel parabolic subgroup P, whereas the cuspidal part of $\mathcal{A}_{short}$ consists precisely of representations which are CAP with respect to the Klingen parabolic Q and the Borel subgroup B. This prediction is easy to see, because given an A-parameter ψ, Arthur's conjecture specifies the near equivalence class of the representations in the global A-packet A_ψ. Thus, we expect that $\mathcal{A}_{SK} = \mathcal{A}_{\rm long,cusp}$ ([13] showed that this is indeed the case).

4 Saito–Kurokawa A-packets

In this section, we examine the fine structure of the Saito–Kurokawa A-packets in greater detail. This will suggest a way of constructing these packets.

4.1 Saito–Kurokawa A-parameters

Recall that we have the subgroup

$$SL_2 \times SL_2 \subset Sp_4.$$

Further, the centralizer of one of these SL_2's is the other SL_2. For a Saito–Kurokawa A-parameter ψ, since $\psi|_{SL_2(\mathbb{C})}$ is an isomorphism onto one of these SL_2, say the second one, $\psi|_{L_F}$ must send L_F into $SL_2 \times \mu_2$, where μ_2 is the center of the second SL_2. Thus to give ψ means to give a (irreducible) map $L_F \to SL_2(\mathbb{C})$ and a quadratic character $L_F \to W_F \to \mu_2$. According to our basic assumption, this means that:
Saito–Kurokawa A-parameters are (conjecturally) parametrized by pairs (τ, χ) where τ is a cuspidal representations of PGL_2 and χ is a quadratic Grossencharacter.

In other words, a typical Saito–Kurokawa parameter looks like:

$$\psi_{\tau,\chi} : L_F \times SL_2(\mathbb{C}) \xrightarrow{\rho_\tau \oplus (\chi \otimes \mathrm{id})} SL_2(\mathbb{C}) \times SL_2(\mathbb{C}) \longrightarrow Sp_4(\mathbb{C}).$$

Observe that for χ fixed, such parameters are indexed by cuspidal representations of PGL_2, just as the near equivalence classes in the space $\widetilde{\mathcal{A}}_{00}$ for $\widetilde{SL}_2$.

Given a parameter as above, it is easy to check that the centralizer (modulo center) $S_{\psi_{\tau,\chi}}$ of $\psi_{\tau,\chi}$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$. Moreover, the local component groups $S_{\psi_{\tau,\chi},v}$ are given by

$$S_{\psi_{\tau,\chi},v} = \begin{cases} 1, & \text{if } \rho_{\tau,v} \text{ is reducible;} \\ \mathbb{Z}/2\mathbb{Z}, & \text{if } \rho_{\tau,v} \text{ is irreducible.} \end{cases}$$

The condition $\rho_{\tau,v}$ is irreducible is equivalent to τ_v being a discrete series representation of $PGL_2(F_v)$.

4.2 Local Arthur packets

Now Arthur's conjecture predicts that for each place v, the local A-packets $A_{\tau,\chi,,v}$ has the form:

$$A_{\tau,\chi,v} = \begin{cases} \{\pi^+_{\tau_v,\chi_v}\}, & \text{if } \tau_v \text{ is not a discrete series;} \\ \{\pi^+_{\tau_v,\chi_v}, \pi^-_{\tau_v,\chi_v}\}, & \text{if } \tau_v \text{ is a discrete series.} \end{cases}$$

Here, $\pi^+_{\tau_v,\chi_v}$ is indexed by the trivial character of $S_{\psi_{\tau,\chi},v}$.

Of course, we know what π_v^+ has to be for almost all v: it is irreducible unramified with Satake parameter $s_{\psi_{\tau_v,\chi_v}}$. This unramifed representation π_v^+ can be alternatively expressed as $J_P(\tau_v, \chi_v, 1/2)$, i.e. the unique irreducible quotient of the generalized principal series unitarily induced from the representation $\tau_v \otimes \chi_v| - |_v^{\frac{1}{2}}$ of the Levi $PGL_2 \times GL_1$ of P. From this, we see that the representations in the global A-packet are nearly equivalent to the constitutents of $\mathrm{Ind}_P^G \tau \otimes \chi| - |^{1/2}$. Thus representations in $\mathcal{A}_{\mathrm{long}}$ are indeed CAP with respect to P.

In fact, one knows what $\pi^+_{\tau_v,\chi_v}$ is in general, for Arthur's conjecture specifies the L-parameter of $\pi^+_{\tau_v,\chi_v}$. In the case at hand, the L-packet for this L-parameter is a singleton. Thus we must have:

$$\pi^+_{\tau_v,\chi_v} \cong J_P(\tau_v, \chi_v, 1/2) \quad \text{for all } v.$$

When we construct the local packets later using theta correspondence, we should check that this is indeed the case for our construction.

The main observation to make here is that (for fixed χ_v) the structure of the local SK A-packet attached to τ_v is identical to the Waldspurger packet of $\widetilde{SL}_2$ associated to τ_v.

4.3 Global A-packets

Let S_τ be the set of places v where τ_v is a discrete series, so that the global A-packet has $2^{\#S_\tau}$ elements. This global packet will contribute to a subspace of L^2_{disc}; we denote the corresponding subspace y $\mathcal{A}(\tau, \chi)$.

To describe the multiplicity of $\pi_\eta \in A_{\tau,\chi}$ in $\mathcal{A}(\tau,\chi)$, we need to know the quadratic character $\epsilon_{\psi_{\tau,\chi}}$ of $S_{\psi_{\tau,\chi}}$. It turns out that $\epsilon_{\psi_{\tau,\chi}}$ is the non-trivial character of $S_{\psi_{\tau,\chi}} \cong \mathbb{Z}/2\mathbb{Z}$ if and only if $\epsilon(\tau \otimes \chi, 1/2) = -1$.

Now if $\pi = \otimes_v \pi^{\epsilon_v}_{\tau_v,\chi_v} \in A_{\tau,\chi}$, then the multiplicity associated to π by Arthur's conjecture is:

$$m(\pi) = \begin{cases} 1, & \text{if } \epsilon_\pi := \prod_v \epsilon_v = \epsilon(\tau \otimes \chi, 1/2); \\ 0, & \text{if } \epsilon_\pi = -\epsilon(\tau \otimes \chi, 1/2). \end{cases}$$

Thus, we should have:

$$\mathcal{A}(\tau, \chi) \cong \bigoplus_{\pi \in A_{\tau,\chi} : \epsilon_\pi = \epsilon(\tau \otimes \chi, 1/2)} \pi.$$

Again, observe that the automorphy of the representations in the global SK A-packet are controlled by the same condition on ϵ-factors as the automorphy of the representations in the global Waldspurger packet associated to τ.

Conclusion 4.1. The above discussion shows that the structure of the space $\widetilde{\mathcal{A}}_{00}$ on $\widetilde{SL}_2$ and the space $\mathcal{A}_{\mathrm{long}}$ on $PGSp_4$ are identical. This should suggest that one can construct the space $\mathcal{A}_{\mathrm{long}}$ by "lifting" from $\widetilde{SL}_2$.

How does one carry out this "lifting"? This can be done using theta correspondence.

5 Construction of Saito–Kurokawa space

Now we are ready to state the results of Piatetski-Shapiro on the space $\mathcal{A}_{SK}$. Consider the dual pair $\widetilde{SL}_2 \times SO(5)$, with $SO(5)$ split. Now to define theta correspondence, it is necessary to specify the quadratic space (V, q) giving rise to $SO(5)$. How many choices are there?

5.1 Odd quadratic spaces and orthogonal groups

Fix a quadratic space V over F. Then the quadratic spaces over F of the same dimension as V are classified by $H^1(F, O(V))$. When $\dim(V)$ is odd, the different forms of $SO(V)$ are classified by $H^1(F, SO(V))$ and we have:

$$H^1(F, O(V)) = H^1(F, SO(V)) \times H^1(F, \mu_2) = H^1(F, SO(V)) \times F^\times/F^{\times 2}.$$

The projection onto $F^\times/F^{\times 2}$ simply gives the discriminant of a quadratic space, whereas the projection onto $H^1(F, SO(V))$ gives the isomorphism class of the associated special orthogonal group. From this, we see that the set of isomorphism classes of quadratic spaces which give rise to a particular special orthogonal group is a principal homogeneous space for $F^\times/F^{\times 2}$ and these quadratic spaces are distinguished by their discriminants.

Given a form G of SO_{2n+1}, there is thus a unique quadratic space of discriminant one whose special orthogonal group is G.

5.2 Twisted theta lifts

In the following, whenever we talk about theta correspondence for $\widetilde{SL}_2 \times G$, it will be defined using this distinguished quadratic space of discriminant one. For convenience, we define the twisted theta lift:

Definition 5.1. *For a quadratic character χ, the χ-twisted theta lift of a representation σ of $\widetilde{SL}_2$ (both locally and globally) is:*

$$\Theta_\chi(\sigma) = \Theta(\sigma) \otimes \chi.$$

5.3 Construction of $\mathcal{A}_{SK}$

By our discussion of the previous section, it is reasonable to expect that $\mathcal{A}_{\mathrm{long}}$ can be constructed by theta lifting from $\widetilde{SL}_2$. More precisely, if we fix a cuspidal representation τ of PGL_2 and a quadratic Grossencharacter χ (which corresponds to an element λ_χ of $F^\times/F^{\times 2}$), then we shall construct the subspace $\mathcal{A}(\tau, \chi)$ of the discrete spectrum by:

$$\mathcal{A}(\tau, \chi) = \Theta_\chi(\widetilde{A}(\tau \otimes \chi)).$$

The question is: is this a reasonable definition?

5.4 Local results

We begin first with local considerations. The local representation $\tau_v \otimes \chi_v$ determines a local Waldspurger packet $\{\sigma_v^+, \sigma_v^-\}$ ($\sigma_v^- = 0$ if τ_v is a principal series). Consider the local theta lifts of the elements of this packet:

$$\pi^{\pm}_{\tau_v,\chi_v} = \Theta_\chi(\sigma_v^{\pm}).$$

These representations are irreducible and unitary. Moreover, all such representations are distinct as τ_v ranges over all unitary representations of $PGL_2(F_v)$ (with χ_v fixed).

Now let A_{τ_v,χ_v} be the local A-packet of $PGSp_4(F_v)$ attached to the Saito–Kurokawa parameter determined by (τ_v, χ_v). Then we *define*:

$$A_{\tau_v,\chi_v} = \{\pi^+_{\tau_v,\chi_v}, \pi^-_{\tau_v,\chi_v}\}.$$

This is a reasonable definition, because we have the following proposition (which verifies directly that Howe's conjecture holds).

Proposition 5.2. *(i)* $\Theta_{\chi_v}(\sigma_v^+) \cong J_P(\tau_v, \chi_v, 1/2)$*;* $\Theta_{\chi_v}(\sigma_v^+)$ *has L-parameter* $\rho_{\tau_v} \oplus \rho_{\chi_v}$.

(ii) When v is archimedean, $\Theta_{\chi_v}(\sigma_v^-)$ can be completely determined (it is a discrete series or a limit of discrete series; we omit the description here but see the example at the end of the section). When v is finite,

$$\Theta_{\chi_v}(\sigma_v^-) = \begin{cases} \textit{supercuspidal, if } \tau_v \otimes \chi_v \textit{ is not Steinberg;} \\ \textit{tempered, if } \tau_v \otimes \chi_v \textit{ is Steinberg.} \end{cases}$$

In fact, when $\tau_v \otimes \chi_v$ is Steinberg, $\Theta_{\chi_v}(\sigma_v^-)$ is the unique non-generic summand in $I_Q(St_{\chi_v})$.

(iii) Moreover, $\Theta_{\chi_v}(\sigma_1) = \Theta_{\chi_v}(\sigma_2)$ implies that $\sigma_1 = \sigma_2$.

The proof of the proposition is an easy exercise, but does require one to know the formulas for the action of the Weil representation $\omega_{\psi,q}$. From (i), we see that the representation $\pi^+_{\tau_v,\chi_v}$ has the expected Langlands parameters required by Arthur's conjecture. The only thing which is not explicit in the proposition is that it does not give some alternative description of $\pi^-_{\tau_v,\chi_v}$; for example, it does not give its Langlands parameter. The main result of Schmidt's paper [19] is the resolution of this local issue:

Proposition 5.3. *The Langlands parameter of $\Theta_{\chi_v}(\sigma_v^-)$ factors through the subgroup $SL_2 \times SL_2$, and as a map into this subgroup, is given by $\rho_{\tau_v} \oplus \rho_{St_{\chi_v}}$. Here, we are using the definition of local L-packets given by Roberts [18]. In particular, this is in agreement with Conjecture 3.2 by Moeglin presented in Section 3.4.*

5.5 Global results

Now suppose that

$$\sigma = \otimes_v \sigma_v^{\epsilon_v} \subset \widetilde{A}(\tau \otimes \chi) \quad \text{with} \quad \prod_v \epsilon_v = \epsilon(\tau \otimes \chi, 1/2).$$

Its χ-twisted global theta lift $V_\chi(\sigma)$ is non-zero (stable range) and one can check that $V_\chi(\sigma)$ is contained in the space of suqare-integrable automorphic forms (i.e. $V_\chi(\sigma) \subset L^2_{\text{disc}}$). Thus, we have:

$$V_\chi(\sigma) \cong \Theta_\chi(\sigma) := \otimes_v \Theta_{\chi_v}(\sigma_v).$$

By the injectivity of local theta correspondence, we know that if $\sigma_1 \ncong \sigma_2$, then $V_\chi(\sigma_1) \ncong V_\chi(\sigma_2)$. Thus, we have:

Proposition 5.4. *Let $A_{\tau,\chi}$ be the global A-packet obtained from the local A-packets A_{τ_v,χ_v} defined above. The χ-twisted global theta lift of $\widetilde{\mathcal{A}}(\tau \otimes \chi)$ is a subspace $\mathcal{A}(\tau, \chi)$ of L^2_{disc} with*

$$\mathcal{A}(\tau, \chi) \cong \bigoplus_{\pi \in A_{\tau,\chi} : \epsilon_\pi = \epsilon(\tau \otimes \chi, 1/2)} \pi.$$

Here $\pi = \otimes_v \pi_{\tau_v,\chi_v}^{\epsilon_v}$ and $\epsilon_\pi = \prod_v \epsilon_v$. Moreover, $\pi = V_\chi(\sigma)$ is contained in the space of cusp forms, unless $L(\tau \otimes \chi, 1/2) \neq 0$ and $\sigma = \sigma^+ := \otimes_v \sigma_v^+$. Moreover, $\mathcal{A}(\tau \otimes \chi, 1) \otimes \chi = \mathcal{A}(\tau, \chi)$.

The proposition says that our definition of the local and global packets leads to a subspace of the discrete spectrum in perfect agreement with Arthur's conjecture. Moreover, the cuspidal π are contained in $\mathcal{A}_{SK}$, since they are CAP with respect to P. In particular, if $\pi \subset \mathcal{A}(\tau, \chi)$, then its partial standard L-functions has the form:

$$L^S(\pi, s) = L^S(\tau, s) \cdot L^S(\chi, s + 1/2) \cdot L^S(\chi, s - 1/2).$$

Thus $L(\pi \otimes \chi, s)$ has a pole at $s = 3/2$. This also shows that the spaces $\mathcal{A}(\tau, \chi)$ have non-isomorphic irreducible constituents (and thus are mutually disjoint), since the standard L-functions are different for different χ. Thus if $\text{Proj}_{\text{cusp}}$ denotes the orthogonal projection onto the cuspidal spectrum, we have:

$$\text{Proj}_{\text{cusp}} \left(\bigoplus_{(\tau,\chi)} \mathcal{A}(\tau, \chi) \right) \subset \mathcal{A}_{SK}.$$

The following theorem is the main result of [13]:

Theorem 5.5.

$$\mathrm{Proj}_{\mathrm{cusp}}\left(\bigoplus_{(\tau,\chi)} \mathcal{A}(\tau,\chi)\right) = \mathcal{A}_{SK}.$$

In fact, the representations in $\mathcal{A}_{SK}$ can be characterized as those cuspidal representations π such that the standard L-function of some quadratic twists of π has a pole somewhere (in which case, a pole must occur at $s = 3/2$).

The proof of this uses the theory of standard L-functions for GSp(4). It is quite intricate and is beyond the scope of these notes. In the last section, we present a proof of a slightly weaker result based on the Rallis inner product formula, the regularized Siegel–Weil formula and the doubling method of Piatetski-Shapiro and Rallis.

A corollary of the theorem is:

Corollary 5.6. *The space $\mathcal{A}_{SK}$ has multiplicity one.*

Proof. This follows from the theorem, the multiplicity-one result for $\widetilde{A}_{00}$, the multiplicity preservation of theta correspondence and the fact that the χ-twists of the space constructed from the theta lifts have non-isomorphic irreducible summands for different χ. □

5.6 Example

We conclude this section with an example. Suppose that f is a classical holomorphic newform of level N and weight $2k$. For no other reason but for simplicity, we assume that N is square-free with S distinct prime factors. Then f corresponds to a cuspidal representation τ_f of PGL_2 such that

$$\tau_{f,v} = \begin{cases} \text{unramified representation, if } v \text{ does not divide } N; \\ \text{the Steinberg representation, if } v \text{ divides } N; \\ \text{holomorphic discrete series of lowest weight } 2k, \text{ if } v \text{ is infinite.} \end{cases}$$

Moreover, $\epsilon(\tau_f, 1/2) = (-1)^S \cdot (-1)^k = (-1)^{S+k}$.

The local Saito–Kurokawa packet associated to τ_f and the trivial (quadratic) character thus has two elements π_v^+ and π_v^- if v is infinite or divides N. For a finite v, we have described these representations above. At the real place, these two representations can be described as the cohomologically induced modules $A_{\mathfrak{q}^\pm}(\lambda_k)$ where $\mathfrak{q}^+$ (resp. $\mathfrak{q}^-$) is the θ-stable Siegel parabolic subalgebra whose Levi subalgebra is the complexification of $U(1,1)$ (resp. $U(2)$) and $\lambda_k = det^{k-2}$. This shows that π_v^- is a holomorphic discrete series when $k \geq 2$ and is a limit of a holomorphic discrete series when $k = 1$.

The global Saito–Kurokawa packet associated to τ_f has 2^{S+1} elements but by the multiplicity formula, only half of them occur in the discrete spectrum.

More precisely, $\pi = \otimes_v \pi_v^{\epsilon_v}$ occurs in the discrete specturm if and only if $\prod_v \epsilon_v = (-1)^{S+k}$.

All these 2^S representations are thus Saito–Kurokawa lifts of f.

To be more classical, suppose we are just interested in Saito–Kurokawa lifts of f which corresponds to Siegel modular forms. Then we should look only at those $\pi = \otimes_v \pi_v^{\epsilon_v}$ with $\epsilon_\infty = -$. If one has a theory of canonical vectors in representations of $PGSp_4$ (i.e., a theory of newforms), then one may pick a distinguished vector in the representation $\pi_v^{\epsilon_v}$. This leads to 2^{S-1} Siegel modular forms which may be considered the *Saito–Kurokawa lifts of* f, when $S \geq 1$. When $S = 0$, i.e., when f is a level one form, then such a Siegel modular form exists if and only if k is odd (this is the classical condition for existence of the Saito–Kurokawa lifts of level 1 forms).

6 Transfer of Saito–Kurokawa representations between inner forms

In this section, we consider the transfer of the Saito–Kurokawa space to the other forms of $PGSp_4$; these are of the form $SO(V,q)$ with $\dim(V) = 5$. Given such a form G', it is natural to construct the transfer of $\mathcal{A}_{SK}$ from $PGSp_4$ to G' via the theta correspondence from $\widetilde{SL}_2$. Thus, given (τ,χ), we want to examine the structure of the χ-twisted theta lift of $\widetilde{\mathcal{A}}(\tau)$ to G'.

If σ is an irreducible constituent of $\widetilde{A}(\tau)$, the global theta lift $\Theta'_\chi(\sigma)$ may now be zero (since we are no longer in the stable range). However, the Rallis inner product formula tells us that it is non-zero if and only if for each v, the local theta lift of σ_v to G'_v is non-zero. Thus, we shall first consider the question of local theta lifts. After that, we can ask: if $\Theta'_\chi(\sigma)$ is non-zero, from which representation of $PGSp_4$ is it lifted (say, in the sense of Langlands parameter)?

6.1 Local theta lifts

Fix a place v. If v is not complex, there is a unique special orthogonal group G'_v whose F_v-rank is 1. Consider the local theta correspondence for $\widetilde{SL}_2(F_v) \times G'_v$. The following can be found in [17]:

Proposition 6.1. *Let σ be an irreducible representation of $\widetilde{SL}_2(F_v)$. Then $\Theta'_{\chi_v}(\sigma) \neq 0$ if and only if* $\mathrm{Hom}_{U(F_v)}(\sigma, \mathbb{C}_{\psi_a}) \neq 0$ *for some $a \notin F_v^{\times 2}$. In fact, the only σ's whose local theta lift is* 0 *are given by:*

(i) if v is finite, then $\sigma_v = \omega_{\psi_v}^{\pm}$;

(ii) if v is real, then σ is a holomorphic discrete series or $\sigma = \omega_{\psi_v}^{\pm}$.

For a finite place, the group G' is the only non-trivial form of $PGSp_4$. On the other hand, when v is real, there is an additional form G_c which is compact and which is defined by a definite quadratic space. For this we have:

Proposition 6.2. *If G' is anisotropic, then $\Theta'(\sigma) \neq 0$ if and only if σ is a holomorphic discrete series of lowest weight $\geq 5/2$.*

6.2 Local packets

We can now define the local packets for the group G'. Suppose we are given a local A-parameter of Saito–Kurokawa type associated to the pair (τ_v, χ_v). We have the local Waldspurger packet $\widetilde{A}_{\tau_v \otimes \chi_v} = \{\sigma_v^+, \sigma_v^-\}$ associated to $\tau_v \otimes \chi_v$ and we consider the set of χ_v-twisted local theta lifts

$$\{\Theta'_{\chi_v}(\sigma_v^+), \Theta'_{\chi_v}(\sigma_v^-)\}.$$

We want to define this as the local packet on G'_v attached to the A-parameter $\psi = \psi_{\tau_v,\chi_v}$. Recall that the representations in the local packet should be indexed by the irreducible characters of Z_ψ/Z_ψ^0 which are non-trivial when restricted to $Z_{\hat{G}'}$.

These two representations are indexed when τ_v is a discrete series, $Z_{\psi_v}/Z^0_{\psi_v} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ in which $Z_{\hat{G}'}$ sits diagonally. There are thus two characters which are non-trivial when restricted to $Z_{\hat{G}'}$, namely η_{+-} and η_{-+} (where η_{+-} is trivial on the first copy of $\mathbb{Z}/2\mathbb{Z}$ and non-trivial on the second copy). We set:

$$\pi^{+-}_{\tau_v,\chi_v} = \Theta'_{\chi_v}(\sigma_v^+) \quad \text{and} \quad \pi^{-+}_{\tau_v,\chi_v} = \Theta'_{\chi_v}(\sigma_v^-).$$

When τ_v is a principal series, $Z_{\psi_v}/Z^0_{\psi_v} \cong \mathbb{Z}/2\mathbb{Z}$, and $Z_{\hat{G}'}$ maps isomorphically onto this. Thus the local packet just consists of the single representation $\Theta'_{\chi_v}(\sigma_v^+)$ which is indexed by the non-trivial character character of $Z_{\psi_v}/Z^0_{\psi_v}$. Now the center of $SL_2 \times SL_2$ lies in Z_{ψ_v} and when restricted to this, the map $Z_{\psi_v} \to Z_{\psi_v}/Z^0_{\psi_v}$ is simply the second projection $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$. Thus the non-trivial character of $Z_{\psi_v}/Z^0_{\psi_v}$ is the character η_{+-} of $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Thus we set

$$\pi^{+-}_{\tau_v,\chi_v} = \Theta'_{\chi_v}(\sigma_v^+).$$

Observe that this labelling is consistent with the discrete series case. This completes our definition of the local packets for the group G'.

For a representation π_v in our packets, we shall set:

$$\epsilon(\pi_v) = \begin{cases} +, & \text{if } \pi_v = \pi_v^{+-}; \\ -, & \text{if } \pi_v = \pi_v^{-+}. \end{cases}$$

The following proposition describes the representations of the packets more explicitly:

Proposition 6.3. *Assume that v is p-adic so that G' has F_v-rank 1.*

(i) If τ_v is the principal series $\pi(\mu_v, \mu_v^{-1})$, then $\pi^{+-}_{\tau_v,\chi_v} = I_{P'}(\chi_v, \mu_v)$; its L-parameter is

$$\rho_{\tau_v} \oplus \rho_{St_{\chi_v}} = \mu_v \oplus \mu_v^{-1} \oplus \rho_{St_{\chi_v}}.$$

(ii) Suppose that τ_v is a discrete series. Then

$$\pi^{-+}_{\tau_v,\chi_v} = J_{P'}(JL(\tau_v), \chi_v, 1/2).$$

Here $JL(\tau_v)$ is the Jacquet–Langlands lift of τ_v to the inner form of PGL_2. The Langlands parameter is $\rho_{\tau_v} \oplus \rho_{\chi_v}$, i.e., the same as the L-parameter of the representation $\pi^{+}_{\tau_v,\chi_v}$ of $PGSp_4(F_v)$. Moreover,

$$\pi^{+-}_{\tau_v,\chi_v} = \begin{cases} \textit{supercuspidal, if } \tau_v \textit{ is supercuspidal;} \\ \textit{discrete series, if } \tau_v \otimes \chi_v \textit{ is twisted Steinberg;} \\ 0, \textit{ if } \tau_v \otimes \chi_v \textit{ is Steinberg.} \end{cases}$$

More precisely, when $\tau_v \otimes \chi_v$ is a twisted Steinberg, $\pi^{+-}_{\tau_v,\chi_v}$ is the irreducible submodule of the induced representation $I_{P'}(JL(\tau_v), \chi_v, 1/2)$.

6.3 Remarks about Moeglin's conjecture

In view of Moeglin's conjecture in Section 3.4, we expect that the L-parameter of $\pi^{+-}_{\tau_v,\chi_v}$ should be equal to $\rho_{\tau_v} \oplus \rho_{St_{\chi_v}}$, i.e. the same as that of $\pi^{-}_{\tau_v,\chi_v}$ for $PGSp_4$. As the above proposition shows, this is the case when τ_v is a principal series and when $\tau_v \otimes \chi_v$ is twisted Steinberg.

When $F_v = \mathbb{R}$, the correspondence can also be (and has been) explicitly determined; we omit the description here, but see [20].

6.4 Remarks

There are two further observations to make here:
(a) As a consequence of (ii), we see that in one particular case, namely when $\tau_v \otimes \chi_v$ is Steinberg, a representation in the local packet is zero even though it is allowed to be non-zero by Arthur's conjecture. Thus the packet has only one element. This is not so surprising. Indeed, we expect the L-parameter of $\pi^{+-}_{St_v,\chi_v}$ to be equal to $\rho_{St_v \otimes \chi_v} \oplus \rho_{St_v \otimes \chi_v}$; but this L-parameter is *irrelevant* since it factors through the Levi of the Siegel parabolic of $Sp_4(\mathbb{C})$.

When $\tau_v \otimes \chi_v$ is not Steinberg, this obstruction is not present; thus the local packet has one or two elements if τ_v is a discrete series, one element otherwise.

Similarly, suppose that $F = \mathbb{R}$ and τ_v is a discrete series. If G'_v has rank 1, we have $\pi^{+-} = 0$. On the other hand, when G'_v is compact, then $\pi^{-+} = 0$, and if further τ_v has lowest weight 2, then $\pi^{+-} = 0$ also.
(b) The Langlands parameter for G' in (i) is not equal to any L-parameter associated to local components of Saito–Kurokawa representations of the split $PGSp_4$. As we shall see later, this implies that there are cuspidal representations in the Saito–Kurokawa space of G' whose Langlands lift to $PGSp_4$ is not contained in the Saito–Kurokawa space of $PGSp_4$ (indeed not contained in the discrete spectrum).

6.5 Global lifting

As in the split case, the subspace of the discrete spectrum corresponding to the A-parameter $\psi_{\tau,\chi}$ can be constructed as:

$$\mathcal{A}_{G'}(\tau,\chi) = V_{G',\chi}(\widetilde{A}(\tau\otimes\chi)).$$

We leave it to the reader to verify that with our definition of local packets, the structure of this representation is in agreement with the multiplicity formula in Arthur's conjecture. In particular, a representation π' with parameter $\psi_{\tau,\chi}$ occurs in the $\mathcal{A}_{G'}(\tau,\chi)$ iff $\prod_v \epsilon(\pi'_v) = \epsilon(\tau\otimes\chi, 1/2)$.

As before, it is clear that

$$\mathrm{Proj}_{\mathrm{cusp}}\left(\bigoplus_{\tau,\chi}\mathcal{A}_{G'}(\tau,\chi)\right) \subset \mathcal{A}_{G',SK}.$$

It is natural to ask if equality holds. The following is the main result of [19]:

Theorem 6.4. *Assume that G' has F-rank 1. Then*

$$\mathrm{Proj}_{\mathrm{cusp}}\left(\bigoplus_{\tau,\chi}\mathcal{A}_{G'}(\tau,\chi)\right) = \mathcal{A}_{G',SK}.$$

The proof of this is similar to that of Piatetski-Shapiro for the split case. It relies on a particular Rankin–Selberg representation of the standard L-function for G'. This Rankin–Selberg integral requires the existence (and uniqueness) of a Bessel functional (i.e. a Fourier coefficient along the unipotent radical of Siegel parabolic). One does not have this if G' is anisotropic, and thus the proof does not work in this case.

However, there is another Rankin–Selberg integral for the standard L-function of G', the so-called doubling method of Piatetski-Shapiro and Rallis [4], which works for all forms G'. Might one be able to show the above theorem for all G' using this? In the next section, we shall attempt to do so, proving a slightly weaker result than the above theorem.

6.6 Some peculiar representations of inner forms

We describe some peculiar cuspidal representations alluded to above in the remarks. The existence of these representations were first noticed by Sayag [19] and this motivated the definition of CAP representations for non-quasi-split groups.

Let us take $G' = SO(V',q')$ to be an inner form of $PGSp_4$. Then there are an even number of places v where G'_v is not split. Call this set of places S. Let τ be a cuspidal representation of PGL_2 and suppose that for some place $v_0 \in S$, τ_{v_0} is a principal series representation $\pi(\mu_{v_0}, \mu_{v_0}^{-1})$.

Let π' be a representation in the global A-packet of G' associated to τ (and the trivial quadratic character) and suppose that $\prod_v \epsilon(\pi'_v) = \epsilon(\tau, 1/2)$, so that π' occurs in $\mathcal{A}_{G',SK}$.

Proposition 6.5. *The Langlands lift of the representation $\pi' \subset \mathcal{A}_{G',SK}$ described above is not contained in the Saito–Kurokawa space $\mathcal{A}_{SK}$ of $PGSp_4$. Moreover, if G' has F-rank 1 (so that the Siegel parabolic P' exists), there are no cuspidal representations τ' for which π' is nearly equivalent to the constituents of $\mathrm{Ind}_{P'}^{G'} \tau'$.*

Proof. The local component π'_{v_0} is the degenerate principal series $I_{P'}(1, \mu_{v_0})$ and its local L-parameter $\rho_{\tau_{v_0}} \oplus \rho_{St_{v_0}}$ is not equal to the L-parameters of local components of representations in the Saito–Kurokawa space $\mathcal{A}_{SK}$ of $PGSp_4$. This confirms the first assertion. For the second, if such a τ' exists, then the Jacquet–Langlands lift of τ' to PGL_2 is nearly equivalent to τ and thus must be equal to τ. But since τ_{v_0} is a principal series, τ_{v_0} is not equal to a Jacquet–Langlands lift. With this contradiction, the second assertion is proved. □

6.7 Remarks

When G' has F-rank 1 and τ is everywhere unramified, the unique π' in the global A-packet is cuspidal and everywhere unramified. In his Ph.D. thesis [14], A. Pitale gave a construction of π' (or rather the unique spherical vector in π') using a converse theorem of Maass. The fact that the representation constructed by Pitale is the same as that constructed here by theta lifting is a consequence of Theorem 6.4 or the weaker Theorem 7.1. The method of construction of π' given in [14] is interesting, as it also gives explicit formulas for the Fourier coefficients of the spherical vector in π'. However, it is not clear how this construction can be extended to the case of general τ and it seems difficult to use this approach to obtain a classification result as refined as the one obtained here.

There is nothing wrong with the fact that the Langlands lift of π' is not contained in $\mathcal{A}_{SK}$. It is not a contradiction to the functoriality conjecture, because the conjecture only requires the Langlands lift to be automorphic; there is no reason why it should be in the discrete spectrum!

A similar phenomenon can already be found in the Jacquet–Langlands correspondence: the Langlands lift of the trivial representation of $PD^\times$ (D a quaternion division algebra) is the anomalous representation of PGL_2 which is Steinberg at the places of ramification of D and trivial everywhere else, and this representation is automorphic but is not contained in the discrete spectrum. However, in the comparison of trace formulas, the constant functions of $PD^\times$ and PGL_2 are matched up, so that there is some reason for considering the trivial representation of PGL_2 as the lift of the trivial representation of $PD^\times$, even though they have different L-parameters. In fact, the correct way to view the Jacquet–Langlands transfer of discrete spectrums is that it is functorial with respect to A-parameters.

Thus, the Saito–Kurokawa space and its transfer to various inner forms are best understood in terms of A-parameters rather than L-parameters.

7 Characterization of Saito–Kurokawa space

In this section, we shall try to show that

$$\mathrm{Proj}_{\mathrm{cusp}}\left(\bigoplus_{\tau,\chi}\mathcal{A}_{G'}(\tau,\chi)\right)=\mathcal{A}_{G',SK}$$

for any inner form G' of $PGSp_4$. However, we shall only be able to prove a weaker result. More precisely, we let

$$\mathcal{B}_{G',SK}\subset\mathcal{A}_{G',SK}$$

be the submodule consisting of representations which are nearly equivalent to the representations in the Saito–Kurokawa A-packets $\bigcup_{\tau,\chi}A_{\tau,\chi}$. Clearly, we have:

$$\mathrm{Proj}_{\mathrm{cusp}}\left(\bigoplus_{\tau,\chi}\mathcal{A}_{G'}(\tau,\chi)\right)\subset\mathcal{B}_{G',SK}.$$

The main result of this section is:

Theorem 7.1. *Let G' be any inner form of $PGSp_4$ (possibly split). Then*

$$\mathrm{Proj}_{\mathrm{cusp}}\left(\bigoplus_{\tau,\chi}\mathcal{A}_{G'}(\tau,\chi)\right)=\mathcal{B}_{G',SK}.$$

Of course when G' is isotropic, this theorem is a consequence of the stronger result of Piatetski-Shapiro [13] and Sayag [19]. In any case, this result shows that our definition of local A-packets is correct.

The proof that we present below is based on the *regularized* Rallis inner product formula (which in turn relies on the regularized Siegel–Weil formula).

Let π be an irreducible summand of $\mathcal{B}_{G',SK}$. Then for some quadratic character χ, $\pi\otimes\chi$ is nearly equivalent to an induced representation $I_P(\tau,1/2)$ of $PGSp_4$, with τ a cuspidal representation of PGL_2. Consider the global theta lift $\widetilde{V}(\pi\otimes\chi)$ of $\pi\otimes\chi$ to $\widetilde{SL}_2$; $\widetilde{V}(\pi\otimes\chi)$ is a subrepresentation of the space of automorphic forms on $\widetilde{SL}_2$. It is not difficult to check that $\widetilde{V}(\pi\otimes\chi)$ is contained in the cuspidal spectrum (we omit the standard computation here). To prove the theorem, we need to show that $\widetilde{V}(\pi\otimes\chi)$ is non-zero.

In the following, we may assume, without loss of generality, that χ is trivial.

7.1 Inner product

For $\varphi_i\in\omega_\psi$ and $f_i\in\pi$, we consider the inner product

$$\langle\widetilde{\theta}(\varphi_1,f_1),\widetilde{\theta}(\varphi_2,f_2)\rangle_{SL_2}$$

where

$$\widetilde{\theta}(\varphi_i, f_i)(h) = \int_{G'_F \backslash G'_{\mathbb{A}}} \theta(\varphi_i)(gh) \cdot \overline{f_i(g)}\, dg.$$

We want to show that this inner product is non-zero for some choice of φ_1 and φ_2 (we fix f_1 and f_2 without loss of generality). Formally exchanging orders of integration, this inner product is equal to

$$\int_{(G'_F \times G'_F)\backslash(G'_{\mathbb{A}} \times G'_{\mathbb{A}})} \overline{f_1(g_1)} f_2(g_2) \cdot \int_{SL_2(F)\backslash SL_2(\mathbb{A})} \theta(\varphi_1)(g_1 h)\overline{\theta(\varphi_2)(g_2 h)}\, dh\, dg_1\, dg_2.$$

However, the inner integral $I(\varphi_1, \varphi_2)$ is not convergent (for general φ_1 and φ_2), so that this exchange is not justified. Fortunately, there is an easy way to regularize this inner integral, as shown by Kudla and Rallis [9]; we explain this regularization next.

7.2 Doubling

Let $\mathbb{V}$ be the ten-dimensional quadratic space $(V', q') \oplus (V', -q')$. Then $\mathbb{V}$ is totally split and $\mathbb{G} = SO(\mathbb{V})$ contains $G' \times G'$ as a natural subgroup. Let W_{ψ_v} be the Weil representation of $\widetilde{SL}_2(F_v) \times \mathbb{G}_v$; it may be realized on the space $S(\mathbb{V}_v)$ of Schwarz functions on $\mathbb{V}_v$ (the Schrödinger model). Then as a representation of $\tilde{S}L_2(F_v) \times (G'_v \times G'_v)$, we have a natural isomorphism:

$$W_{\psi_v} \cong \omega_{\psi_v, q} \otimes \omega_{\psi_v, -q},$$

with the tensor on the right being an inner tensor for the $\widetilde{SL}_2$ action and an outer tensor for the action of $G'_v \times G'_v$. This isomorphism is induced by the natural isomorphism of vector spaces:

$$S(V'_v) \hat{\otimes} S(V'_v) \cong S(\mathbb{V}_v).$$

Note in particular that the representation W_ψ factors to the linear group SL_2.

Similarly, on the global level, the natural isomorphism

$$\iota : S(V'_{\mathbb{A}}) \hat{\otimes} S(V'_{\mathbb{A}}) \cong S(\mathbb{V}_{\mathbb{A}})$$

gives an isomorphism $\omega_{\psi,q} \otimes \omega_{\psi,-q} \cong W_\psi$ as representations of $\widetilde{SL}_2(\mathbb{A}) \times (G'_{\mathbb{A}} \times G'_{\mathbb{A}})$. Moreover, we have:

$$\Theta(\iota(\varphi_1 \otimes \varphi_2)((g_1, g_2)h) = \theta(\varphi_1)(g_1 h) \cdot \overline{\theta(\varphi_2)(g_2 h)}.$$

Thus, the integral that we need to regularize is equal to

$$I(\Phi)(g_1, g_2) = \int_{SL_2(F)\backslash SL_2(\mathbb{A})} \Theta(\Phi)((g_1, g_2)h) dh$$

for $\Phi \in S(\mathbb{V}_{\mathbb{A}})$. This is simply the theta lift of the constant function of SL_2 to $\mathbb{G}_{\mathbb{A}}$ (restricted to $G'_{\mathbb{A}} \times G'_{\mathbb{A}}$) if it were convergent.

7.3 Regularization

It turns out that $I(\Phi)$ converges absolutely if and only if $W_\psi(h)\Phi(0)$ is zero for all $h \in SL_2(\mathbb{A})$. The map which associates to Φ the function

$$h \mapsto W_\psi(h)\Phi(0)$$

is an $SL_2(\mathbb{A})$-intertwining map

$$T : W_\psi \longrightarrow \text{a principal series of } SL_2(\mathbb{A})$$

(this map is also $\mathbb{G}_\mathbb{A}$-invariant). Thus, $I(\Phi)$ is absolutely convergent if and only if Φ lies in the kernel of T.

Now, at any archimedean place v_0, this principal series is easily seen to have different infinitesimal character from the trivial representation of SL_2. Thus one can find an element Z in the center of the universal enveloping algebra of $\mathfrak{sl}_2(F_{v_0})$ with the property that

$$Z = \begin{cases} 1, \text{ on the trivial representation;} \\ 0, \text{ on the above principal series.} \end{cases}$$

Then for any $\Phi \in S(\mathbb{V}_\mathbb{A})$, $W_\psi(Z)\Phi$ lies in $ker(T)$ and the regularization of $I(\Phi)$ is defined to be:

$$I^{reg}(\Phi) = I(W_\psi(Z)(\Phi)).$$

The integral $I^{reg}(\Phi)$ is the regularized theta lift of the constant function of SL_2 to $\mathbb{G}$. Indeed, because the action of Z commutes with both SL_2 and $\mathbb{G}$, the map $\Phi \mapsto I^{reg}(\Phi)$ is both $SL_2(\mathbb{A})$-invariant and $\mathbb{G}_\mathbb{A}$-intertwining, just as the map $\Phi \mapsto I(\Phi)$ would be if it were convergent. Moreover, if $\Phi \in \ker(T)$ to begin with, then

$$I^{reg}(\Phi) = I(\Phi).$$

Though this regularization may seem somewhat ad-hoc, as it involves the choice of Z, it is in fact canonical, for one can show that there is at most one extension of I from $\ker(T)$ to W_ψ which is both SL_2-invariant and $\mathbb{G}$-intertwining.

7.4 Regularized inner product

How is this regularized integral relevant to our problem? Let us consider the absolutely convergent integral

$$\int_{(G'_F \times G'_F)\backslash(G'_\mathbb{A} \times G'_\mathbb{A})} \overline{f_1(g_1)} f_2(g_2) \cdot \int_{SL_2(F)\backslash SL_2(\mathbb{A})} \Theta\left(W_\psi(Z)(\iota(\varphi_1 \otimes \varphi_2))\right)((g_1, g_2)h)\, dh\, dg_1\, dg_2.$$

Now we can write

$$W_\psi(Z)(\iota(\varphi_1 \otimes \varphi_2)) = \sum_{k=1}^{r} \iota(\varphi_{1,k} \otimes \varphi_{2,k}).$$

Thus, on exchanging orders of integration, we see that the above integral is equal to

$$\sum_{k=1}^{r} \langle \widetilde{\theta}(\varphi_{1,k}, f_1, \widetilde{\theta}(\varphi_{2,k}, f_2)\rangle_{SL_2}.$$

In other words, to prove the theorem, it remains to show that

$$\int_{(G'_F \times G'_F)\backslash(G'_{\mathbb{A}} \times G'_{\mathbb{A}})} \overline{f_1(g_1)} \cdot f_2(g_2) \cdot I^{reg}(\Phi)(g_1, g_2)\, dg_1\, dg_2$$

is non-zero for some choice of Φ.

7.5 Regularized Siegel–Weil formula

The regularized integral $I^{\mathrm{reg}}(\Phi)$ turns out to be equal to the residue of an Eisenstein series on $\mathbb{G}_{\mathbb{A}}$; this is a special case of the regularized Siegel–Weil formula. More precisley, there is a natural $SL_2(\mathbb{A})$-invariant and $\mathbb{G}_{\mathbb{A}}$-equivariant map

$$R : W_\psi \longrightarrow \mathrm{Ind}_{\mathbb{P}'}^{\mathbb{G}} \delta_{\mathbb{P}'}^{1/4} \quad \text{(unnormalized induction)}$$

where $\mathbb{P}'$ is a maximal parabolic subgroup of $\mathbb{G}$ stabilizing a maximal isotropic subspace X of $\mathbb{V}$. We note here that in $\mathbb{G}$, there are two conjugacy classes of maximal isotropic subspaces and our discussion below does not depend on the choice of the class of the maximal isotropic subspace X.

The definition of this map R is analogous to that of the map T above, but involves a mixed model for W_ψ rather than the Schrödinger model used in the definition of T. Indeed, the Weil representation W_ψ can also be realized on $S(W \otimes X^*)$ (W is standard two-dimensional symplectic space), and the isomorphism

$$S(\mathbb{V}) \longrightarrow S(W \otimes X^*)$$

is given by a partial Fourier transform: $\Phi \mapsto \hat{\Phi}$. For $\hat{\Phi} \in S(W \otimes X^*)$,

$$R(\hat{\Phi})(g) = (W_\psi(g)\hat{\Phi})(0).$$

Moreover, we have:

Proposition 7.2. *The degenerate principal series* $\mathrm{Ind}_{\mathbb{P}'}^{\mathbb{G}} \delta_{\mathbb{P}'}^{1/4}$ *has a unique irreducible submodule. This irreducible submodule* Π *is spherical and is the so-called minimal representation of* $\mathbb{G}_{\mathbb{A}}$*. The map* R *gives an* $\mathbb{G}_{\mathbb{A}}$*-equivariant isomorphism*

$$(W_\psi)_{SL_2(\mathbb{A})} \cong \Pi \subset \mathrm{Ind}_{\mathbb{P}'}^{\mathbb{G}} \delta_{\mathbb{P}'}^{1/4}.$$

Now each $K_{\mathbb{G}}$-finite $f' \in \operatorname{Ind}_{\mathbb{P}'}^{\mathbb{G}} \delta_{\mathbb{P}'}^{1/4}$ determines a standard section f'_s of the family of degenerate principal series $\operatorname{Ind}_{\mathbb{P}'}^{\mathbb{G}} \delta_{\mathbb{P}'}^{s}$, and we may form the Eisenstein series $E(f', s, g)$. It turns out that at $s = 1/4$, $E(f', s, g)$ is holomorphic, and the map

$$f' \mapsto E(f', s, g),$$

initially defined for $K_{\mathbb{G}}$-finite f', extends to give a $\mathbb{G}_{\mathbb{A}}$-equivariant map from Π to the space of square-integrable automorphic forms on $\mathbb{G}_{\mathbb{A}}$. The *regularized Siegel–Weil formula* states:

Proposition 7.3. *There is a non-zero constant c such that*

$$I^{reg}(\Phi)(g) = c \cdot E(R(\hat{\Phi}), 1/4, g).$$

There is no doubt that the constant c can be precisely determined, but we will not need its precise value here.

7.6 Intertwining operator

It turns out to be more convenient to work with the degenerate principal series at $s = 3/4$ rather than at $s = 1/4$. Let $\mathbb{P}$ be the standard parabolic subgroup which is associated with the other conjugacy class of maximal isotropic subspaces of $\mathbb{V}$. Thus, $\mathbb{P}$ and $\mathbb{P}'$ are associates but not conjugate. There is a standard intertwining map

$$M(s) : \operatorname{Ind}_{\mathbb{P}}^{\mathbb{G}} \delta_{\mathbb{P}}^{1/2+s} \longrightarrow \operatorname{Ind}_{\mathbb{P}'}^{\mathbb{G}} \delta_{\mathbb{P}'}^{1/2-s}$$

and $M(s)$ is meromorphic in s. At $s = 1/4$, $M(s)$ has a pole of order 1 and so we have

$$M = \operatorname{Res}_{s=1/4} M(s) : \operatorname{Ind}_{\mathbb{P}}^{\mathbb{G}} \delta_{\mathbb{P}}^{3/4} \longrightarrow \operatorname{Ind}_{\mathbb{P}'}^{\mathbb{G}} \delta_{\mathbb{P}'}^{1/4}.$$

The image of this map is precisely equal to Π. Moreover, by the functional equation for Eisenstein series,

$$\operatorname{Res}_{s=3/4} E(f, s, g) = E(M(f), 1/4, g).$$

Together with the regularized Siegel–Weil formula, this gives:

Corollary 7.4. *As f varies over all elements of $\operatorname{Ind}_{\mathbb{P}}^{\mathbb{G}} \delta_{\mathbb{P}}^{3/4}$, the space of functions $\operatorname{Res}_{s=3/4} E(f, s, g)$ is the same as the space of functions $I^{reg}(\Phi)(g)$ as Φ ranges over $S(\mathbb{V}_{\mathbb{A}})$. In particular, to prove the theorem, it suffices to show that*

$$\int_{(G'_F \times G'_F)\backslash(G'_{\mathbb{A}} \times G'_{\mathbb{A}})} \overline{f_1(g_1)} f_2(g_2) \cdot E(f, s, (g_1, g_2))\, dg_1\, dg_2$$

has a pole at $s = 3/4$ for some $K_{\mathbb{G}}$-finite $f \in \operatorname{Ind}_{\mathbb{P}}^{\mathbb{G}} \delta_{\mathbb{P}}^{3/4}$.

7.7 Standard L-functions

Piatetski-Shapiro and Rallis showed in [4] that the expression in the above corollary is a Rankin–Selberg representation for the standard L-function of π. To state their result, let us fix the maximal isotropic subspace

$$\Delta V' = \{(v,v) \in \mathbb{V} : v \in V'\}$$

and suppose without loss of generality that $\mathbb{P}$ is the stabilizer of $\Delta V'$. Then the action of $G' \times G'$ on $\mathbb{P}\backslash\mathbb{G}$ has a unique Zariski open dense orbit, namely the orbit of $\Delta V'$. The stabilizer of $\Delta V'$ is equal to $\Delta G'$ and thus $G' \times \{1\}$ acts simply transitively on an open dense subset of $\mathbb{P}\backslash\mathbb{G}$. Now we have:

Proposition 7.5. *For $\Re(s)$ sufficiently large and for a sufficiently large finite set S of places of F,*

$$\int_{(G'_F \times G'_F)\backslash(G'_{\mathbb{A}} \times G'_{\mathbb{A}})} \overline{f_1(g_1)} f_2(g_2) \cdot E(f,s,(g_1,g_2))\, dg_1\, dg_2$$
$$= \prod_{v \in S} Z_v(f_{1,v}, f_{2,v}, f_v, s) \cdot L^S(\pi, 4s-3/2) \cdot \frac{1}{\zeta^S(8s)\cdot\zeta^S(8s-2)}.$$

Here, for $v \in S$,

$$Z_v(f_{1,v}, f_{2,v}, f_v, s) = \int_{G'_v} \langle \pi_v(g)(f_{1,v}), f_{2,v}\rangle \cdot f_{v,s}(g,1)\, dg$$

where $\langle -,-\rangle$ is the G'_v-invariant inner product on π_v which is linear in the second variable and conjugate-linear in the first.

Now the left-hand side of the identity in the above proposition extends to a meromorphic function of s if f is $K_{\mathbb{G}}$-finite. On the other hand, for the case at hand, we know a priori that

$$L^S(\pi,s) = \zeta^S(s+1/2)\cdot\zeta^S(s-1/2)\cdot L^S(\tau,s)$$

for some cuspidal representation of PGL_2 and thus $L^S(\pi,1/2)$ has a meromorphic continuation to $\mathbb{C}$. Finally, the local zeta factors Z_v (for $v \in S$) have been studied in detail in [8]. It was shown in [8, Theorem 3.2.2], that for any smooth f, $Z_v(f_{1,v}, f_{2,v}, f_v, s)$ converges for $\Re(s)$ large and has meromorphic continuation to all of $\mathbb{C}$. Thus, the identity in the proposition holds for all $s \in \mathbb{C}$ and is an equality of meromorphic functions, as long as f is $K_{\mathbb{G}}$-finite.

7.8 End of proof

Now we are given that $L^S(\pi, 4s-3/2))$ has a pole at $s = 3/4$. Thus, by the above proposition, the proof of the theorem is reduced to:

Lemma 7.6. *For each $v \in S$, there exists a $K_{\mathbb{G}_v}$-finite f_v so that $Z_v(f_{1,v}, f_{2,v}, f_v, s)$ is non-zero at $s = 3/4$.*

Proof. Let ϕ be an arbitrary function in $C_c^\infty(G'_v)$. Define a function

$$f_\phi \in \operatorname{Ind}_{\mathbb{P}_v}^{\mathbb{G}_v} \delta_{\mathbb{P}}^{3/4}$$

by requiring that f_ϕ vanishes outside the set $\mathbb{P}_v \cdot (G'_v \times \{1\})$ and

$$F_\phi(p \cdot (g, 1)) = \delta_{\mathbb{P}}(p)^{3/4} \cdot \phi(g).$$

This is well-defined because $\mathbb{P}_v \cdot (G'_v \times \{1\})$ is open dense in $\mathbb{G}_v$ and is analytically isomorphic to $\mathbb{P}_v \times G'_v$.

Let $f_{\phi,s}$ be the standard section extending f_ϕ. Then it is easy to see that the integral which defines $Z(f_{1,v}, f_{2,v}, f_\phi, s)$ for $\Re(s) \gg 0$ is in fact convergent for all s. At $s = 3/4$, it is equal to:

$$\int_{G'_v} \langle \pi_v(g)(f_{1,v}), f_{2,v} \rangle \cdot \phi(g)\, dg.$$

Now since $\langle \pi_v(g) f_{1,v}, f_{2,v} \rangle$ is a non-zero function of g, it is clear that one can find ϕ such that this integral is non-zero.

Now if v is finite, then f_ϕ is already $K_{\mathbb{G}_v}$-finite and so we are done. When v is archimedean, f_ϕ may not be $K_{\mathbb{G}_v}$-finite, but [8] shows that $Z_v(f_{1,v}, f_{2,v}, f_v, s)$ is continuous in f_v for fixed s. Thus the lemma follows by the density of $K_{\mathbb{G}_v}$-finite vectors. □

The theorem is proved.

7.9 Remarks

(i) It seems likely that the above argument can be pushed to yield the theorem with $\mathcal{B}_{G',SK}$ replaced by $\mathcal{A}_{G',SK}$.

(ii) What the proof shows is that if π is a cuspidal representation of G' such that for some quadratic character χ, $L^S(\pi \otimes \chi, s)$ has a pole at $s = 3/2$, then $\pi \otimes \chi$ has non-zero theta lift to $\widetilde{SL}_2$ and thus

$$\pi \subset \operatorname{Proj}_{\text{cusp}} \left(\bigoplus_{\tau,\chi} \mathcal{A}_{G'}(\tau, \chi) \right).$$

Finally, we note the following corollary of the theorem:

Corollary 7.7. *Let π be a representation in a Saito–Kurokawa packet for G'. Then $m_{\text{cusp}}(\pi) \leq 1$.*

This corollary puts one in a position to apply the result of C. Sorensen [21] about level-raising congruences for Saito–Kurokawa representations of G'. We finish up with an example which will be relevant for the application of the main theorem of [21].

7.10 Example

Suppose f is a holomorphic cuspidal newform of weight 4 with respect to $\Gamma_0(N)$ with $N = p_1, \ldots, p_r$ squarefree. The corresponding cuspidal representation τ_f of PGL_2 is unramified outside $S_f = \{p_1, \ldots, p_r, \infty\}$, Steinberg at p_i and a discrete series of lowest weight 4 at the archimedean place. Suppose that r is odd, so that $\epsilon(\tau_f, 1/2) = (-1)^r = -1$. The Saito–Kurokawa packet determined by τ_f (and the trivial character) has 2^{r+1} representations of which 2^r occur in the discrete spectrum. One of these is the representation $\pi_f = \pi_\infty^- \otimes (\otimes_p \pi_p^+)$, with π_∞^- a holomorphic discrete series and π_p non-tempered for all p. This π occurs in the discrete spectrum because of the odd number of minus signs.

Let G' be the inner form of $PGSp_4$ which is ramified precisely at $S_f = \{p_1, \ldots, p_r, \infty\}$ with $G'(\mathbb{R})$ compact. Now there is only one representation π' in the Saito–Kurokawa packet of $G'(\mathbb{A})$ corresponding to τ_f. This is the theta lift of the representation

$$\sigma = \sigma_\infty^+ \otimes (\otimes_{p \in S_f} \sigma_p^-) \otimes (\otimes_{p \notin S_f} \sigma_p^+)$$

which is contained in $\widetilde{A}_{00}$ because of the odd number of minus signs. Thus π' occurs in the discrete spectrum of G' and has the following properties:

- its archimedean component π'_∞ is the trivial representation of $G'(\mathbb{R})$;
- it is the Langlands lift of π, i.e. for all v, π'_v and π_v have the same L-parameters;
- its multiplicity in $L^2(G'_\mathbb{Q} \backslash G'_\mathbb{A})$ is equal to 1.

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Values of Archimedean Zeta Integrals for Unitary Groups

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Summary. We prove that certain archimedean integrals arising in global zeta integrals involving holomorphic discrete series on unitary groups are predictable powers of π times rational or algebraic numbers. In some cases we can compute the integral exactly in terms of values of gamma functions, and it is plausible that the value in the most general case is given by the corresponding expression. Non-vanishing of the algebraic factor is readily demonstrated via the explicit expression.

Regarding analytical aspects of such integrals, whether archimedean or p-adic, a recent systematic treatment is [Lapid–Rallis 2005] in the Rallis conference volume. In particular, the results of Lapid and Rallis allow us to focus on the arithmetic aspects of the special values of the integrals. This is implicit in (4.4)(iv) in [Harris 2006].

Roughly, the integrals here are those that arise in the so-called *doubling* method if the Siegel-type Eisenstein series is differentiated *transversally* before being restricted to the smaller group. In traditional settings, details involving Fourier expansions would be apparent, but such details are inessential. Specifically, many Fourier-expansion details concerning classical Maaß–Shimura operators are spurious, referring, in fact, only to the structure of holomorphic discrete series representations. Of course, the translation to and from rationality issues in spaces of automorphic forms should not be taken lightly.

1 Introduction

We are interested in evaluating archimedean local integrals arising from restricting Siegel-type Eisenstein series from a maximally rationally split unitary group $U(n,n)$ to a product $U(\Phi) \times U(-\Phi)$ of smaller unitary groups, and integrating against $f \otimes f^\vee$ for a holomorphic cuspform on $U(\Phi)$, where $f^\vee$ is the complex conjugated function on $U(-\Phi)$. In the (rational) double coset space

$$\text{(Siegel parabolic in } U(n,n)) \setminus U(n,n) \;/\; (U(\Phi) \times U(-\Phi))$$

there is just one double coset whose contribution to the integral (against a cuspform) is non-zero, and the corresponding global integral *unwinds* to an integral which can be viewed as a product of local integral operators. We arrive at the following situation.

From a global situation in which a unitary group $G = U(\phi)$ is defined via a totally real number field E_o and a CM extension E, we fix throughout this discussion a real imbedding of E_o and a complex imbedding of E, and identify these fields with their images. Then G becomes a unitary group which over E conjugate (by the inertia theorem for Hermitian forms) over E to the unitary group of the Hermitian form

$$H = \begin{pmatrix} -1_p & 0 \\ 0 & 1_q \end{pmatrix}$$

with $p \geq q > 0$. We use this copy of the group, with the side effect that all rationality claims can be made only over E, not $\mathbb{Q}$, for example. Make the usual choice

$$K = \{\begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \in G\} \approx U(p) \times U(q)$$

of maximal compact subgroup of G. Let $\mathfrak{g}$ be the real Lie algebra of G (literally, over E_o, etc.), with complexification $\mathfrak{g}_{\mathbb{C}}$, and as usual

$$\mathfrak{p}_+ = \{\begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix} \in \mathfrak{g}_{\mathbb{C}}\} \quad \mathfrak{p}_- = \{\begin{pmatrix} 0 & 0 \\ * & 0 \end{pmatrix} \in \mathfrak{g}_{\mathbb{C}}\}.$$

The Harish-Chandra decomposition is

$$G \subset N_+ \cdot K_{\mathbb{C}} \cdot N_- \subset G_{\mathbb{C}}$$

with $N_\pm = \exp \mathfrak{p}_\pm$. With minimal parabolic

$$B_K = \{\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} : A \text{ and } B \text{ lower-triangular}\}$$

in $K_{\mathbb{C}}$, the group

$$B = B_K \cdot N_- = \{\text{lower triangular } g \in G_{\mathbb{C}} \approx GL(p+q, \mathbb{C})\}$$

is a minimal parabolic in $G_{\mathbb{C}}$. Further, there is an open subset Ω of N_+ such that

$$G \cdot B = G \cdot K_{\mathbb{C}} \cdot N_- = \Omega \cdot K_{\mathbb{C}} \cdot N_-.$$

The irreducible τ of K extends to a holomorphic irreducible of $K_{\mathbb{C}}$, and this extension has extreme weight $\xi = \xi_\tau$, a one-dimensional representation of the minimal parabolic B_K in $B_{\mathbb{C}}$. Extend ξ to a one-dimensional representation of B by making it trivial on N_-. Harish-Chandra described the dual $\pi_\tau^\vee$ of the holomorphic discrete series π_τ with extreme K-type τ as a collection of functions φ on G which extend holomorphically to an open subset of the complexification of G:

$$\pi_\tau^\vee = \{\text{holomorphic } \varphi \text{ on } GB : \varphi \in L^2(G), \varphi(xb) = \xi(b)^{-1} \cdot \varphi(x), x \in GB, b \in B\}$$

with the *left* regular representation

$$L_h\varphi(g) = \varphi(h^{-1}g).$$

Since π_τ is in the discrete series, there is a single copy of $\pi_\tau \otimes \pi_\tau^\vee$ inside $L^2(G)$, with the biregular representation of $G \times G$ on $L^2(G)$ (where the first factor of G acts by the right regular representation and the second factor acts by the left regular representation). In particular, the copy of $\tau \otimes \pi_\tau^\vee$ in $L^2(G)$ (where τ is the extreme K-type in π_τ) is

$$\tau \otimes \pi_\tau^\vee = \left\{ \text{holomorphic } \varphi \text{ on } G \cdot B : \begin{array}{c} \varphi \in L^2(G), \\ \varphi(xb) = \xi(b)^{-1} \cdot \varphi(x), x \in GB, b \in B, \\ \varphi \text{ left annihilated by } \mathfrak{p}_+ \end{array} \right\}$$

since the right (N_-)-invariance certainly implies right annihilation by $\mathfrak{p}_-$. Then the copy $\tau \otimes \tau^\vee$ of the tensor product of the extreme K-types in $\pi_\tau \otimes \pi_\tau^\vee$ consists of functions which, further, are left annihilated by $\mathfrak{p}_+$, namely

$$\tau \otimes \tau^\vee = \left\{ \text{holomorphic } \varphi \text{ on } G \cdot B : \begin{array}{l} \varphi(xb) = \xi(b)^{-1} \cdot \varphi(x), x \in GB, b \in B, \\ \varphi \text{ right annihilated by } \mathfrak{p}_-, \\ \quad \text{and left annihilated by } \mathfrak{p}_+ \end{array} \right\}.$$

Since these functions φ are holomorphic on the non-empty open subset GB of $G_\mathbb{C}$, the left annihilation by $\mathfrak{p}_+$ implies that φ extends to a left (N_+)-invariant and right (N_-)-invariant holomorphic function on

$$N_+ \cdot K_\mathbb{C} \cdot N_-.$$

Thus,

$$\tau \otimes \tau^\vee = \left\{ \text{holomorphic } \varphi \text{ on } N_+ \cdot K_\mathbb{C} \cdot N_- : \begin{array}{l} \varphi(xb) = \xi(b)^{-1} \cdot \varphi(x), x \in GB, b \in B, \\ \varphi \text{ left } (N+-)\text{-invariant} \end{array} \right\}.$$

Thus, any function φ in this copy of $\tau \otimes \tau^\vee$ extends to $N + -K_\mathbb{C}N_-$, is completely determined by its values on $K_\mathbb{C}$, and on $K_\mathbb{C}$ the function φ is holomorphic and lies in the (holomorphic extension of the) unique copy of $\tau \otimes \tau^\vee$ inside $L^2(K)$. The latter consists of (holomorphic extensions of) coefficient functions $c_{u,v}$ for $u, v \in \tau$, where as usual the matrix coefficient function is

$$c_{u,v}(\theta) = \langle \tau(\theta)u, v \rangle$$

where $\langle\, , \rangle$ is a fixed K-invariant Hermitian inner product on τ. We can assume that for g in the complexification $K_\mathbb{C}$

$$\tau(g^*) = \tau(g)^*$$

where $g \longrightarrow g^*$ is the involution on $K_\mathbb{C}$ which fixes the real points K of $K_\mathbb{C}$.

The integral of interest is of the form

$$Tf(g) = \int_G f(h)\,\overline{\eta(g^{-1}h)}\,dh$$

with f and η as follows. It is important to understand that this integral may not be absolutely convergent, but is defined by analytic continuation, which is known to exist by the result of Lapid and Rallis cited above. We use an analytic continuation (below) convenient for appraisal of rationality properties.

Anticipating insertion of a convergence factor below, rewrite the integral (replacing h by gh) as

$$Tf(g) = \int_G f(gh)\,\overline{\eta(h)}\,dh.$$

For us, f is a *bounded* [1] function on G, is right annihilated by $\mathfrak{p}_-$, has right K-type τ, and under the right regular representation generates (a *copy* of) the holomorphic discrete series representation π_τ of G with extreme K-type τ. [2]

The function η is in $L^2(G)$, is annihilated on the left by $\mathfrak{p}_+$, is annihilated on the right by $\mathfrak{p}_-$, and has right K-type τ. For lowest K-type τ of sufficiently high extreme weight, the universal $(\mathfrak{g}, K)$-module generated by a vector v_τ of K-type τ and annihilated by $\mathfrak{p}_-$ is *irreducible.* [3] Thus, η generates the holomorphic discrete series $\tau \otimes \tau^\vee \subset L^2(G)$ and is a finite sum of functions made from holomorphic extensions of the coefficient function of τ, namely functions of the form

$$\eta_{x,y}(n_+\theta n_-) = c_{x,y}(\theta) = \langle \tau(\theta)x, y\rangle$$

with $x, y \in \tau$.

The way that η arises in practice allows some control over its choice inside the extreme K-type $\tau \otimes \tau^\vee$ inside $\pi \otimes \pi^\vee$. Since $\tau \otimes \tau^\vee$ occurs just once inside $\pi \otimes \pi^\vee$, the space of K-conjugation invariant functions in $\tau \otimes \tau^\vee$ is one-dimensional. Take K-conjugation invariant η defined by

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[1]This boundedness is a crude but sufficient description of the asymptotic behavior of f. In fact, f comes from a holomorphic cuspform, so may have somewhat better decay properties than mere boundedness.

[2]Thus, while f generates a unitary representation, this neither implies nor assumes that f is in $L^2(G)$. Indeed, in the case of interest, f is definitely *not* square-integrable.

[3]This has been known at least since H. Rossi and M. Vergne, *Representations of certain solvable Lie groups on Hilbert spaces of holomorphic functions, and the application to the holomorphic discrete series of a semisimple Lie group*, J. of Func. An. **13** (1973), pp. 324–389, and *Analytic continuation of the holomorphic discrete series of a semi-simple Lie group*, Acta Math. **136** (1976), pp. 1–59. Also, a direct proof of this irreducibility follows easily from consideration of eigenvalues of the Casimir operator on generalized Verma modules.

$$\eta(n_+\theta n_-) = \dim\tau\cdot\mathrm{tr}\tau(k) = \dim\tau\cdot\sum_i\langle\tau(\theta)x_i, x_i\rangle \in \tau\otimes\tau^\vee \subset \pi\otimes\pi^\vee \subset L^2(G)$$

where $\{x_i\}$ is an orthonormal basis for τ. Note that the expression for η does not depend upon the choice of orthonormal basis, but does present the holomorphic extension to $K_\mathbb{C}$ of the character $\mathrm{tr}\tau$ as a function in $\tau\otimes\tau^\vee\subset\pi\otimes\pi^\vee$. Recall that

$$\int_K \mathrm{tr}\tau(k)\cdot k\cdot v\,dk = \frac{v}{\dim\tau}$$

for a vector v in any copy of τ, from Schur orthogonality relations.

Normalize a measure on G as follows. Use a Cartan decomposition

$$G = C\cdot K \approx C\times K$$

where

$$C = \{g\in G = U(p,q) : g = g^* \text{ is positive-definite Hermitian}\}.$$

Give K a Haar measure with total mass 1. Parametrize C by

$$D_{p,q}\ni z \longrightarrow h_z = \begin{pmatrix} (1_p - zz^*)^{-1/2} & z(1_q - z^*z)^{-1/2} \\ (1_q - z^*z)^{-1/2}z^* & (1_q - z^*z)^{-1/2}\end{pmatrix}$$

where $D_{p,q}$ is the classical domain

$$D_{p,q} = \{z = p\text{-by-}q \text{ complex matrix} : 1_p - zz^* \text{ is positive-definite }\}.$$

The unitary group $U(p,q)$ acts on $D_{p,q}$ by generalized linear fractional transformations, and has invariant measure on $D_{p,q}$ which we normalize to

$$d^*z = \frac{dz}{\det(1_q - z^*z)^{p+q}} = \frac{dz}{\det(1_p - zz^*)^{p+q}}$$

where dz is the product of usual additive Haar measures.

2 Qualitative theorem

In the situation described above,

Theorem 2.1. *With f of right K-type τ in a holomorphic discrete series representation $\pi = \pi_\tau$ with lowest K-type τ, and with η as above,*

$$(Tf)(g) = \int_G f(gh)\,\overline{\eta(h)}\,dh = \pi^{pq}\cdot (E\text{-rational number})\cdot f(g)$$

Proof. For computational purposes, we would *like* f to be in $L^2(G)$, allowing use of the Harish-Chandra decomposition as for η. However, f is unlikely to be in $L^2(G)$, despite the fact that it generates a (unitary) representation isomorphic to a discrete series representation. Still, *if* $\eta \in L^1(G)$, then the map $f \longrightarrow Tf$ would be a continuous endomorphism of the representation space generated by f, justifying computation of the integral in any preferred model for π_τ. Despite the fact that, for lowest K-type τ *too low*, η is not in $L^1(G)$, η can be modified by a *convergence factor* parametrized by $s \in \mathbb{C}$ (normalized to be trivial at $s = 0$). This convergence factor is *bounded* for $\Re(s) \geq 0$, is 1 at $s = 0$, and makes the modified η integrable for $\Re(s)$ sufficiently large. The resulting integral has an analytic continuation back to the point $s = 0$, and is evaluated there. [4] The convergence factor is left and right K-invariant, so the K-conjugation invariance of the original η is not disturbed.

A Cartan decomposition $h = h_z k$ and Harish-Chandra decomposition $h_z = n_z^+ \theta_z n_z^-$ are

$$h_z = \begin{pmatrix} (1_p - zz^*)^{-1/2} & z(1_q - z^*z)^{-1/2} \\ (1_q - z^*z)^{-1/2} z^* & (1_q - z^*z)^{-1/2} \end{pmatrix}$$

$$= \begin{pmatrix} 1_p & z \\ 0 & 1_q \end{pmatrix} \begin{pmatrix} (1_p - zz^*)^{1/2} & 0 \\ 0 & (1_q - z^*z)^{-1/2} \end{pmatrix} \begin{pmatrix} 1_q & 0 \\ z^* & 1_q \end{pmatrix}.$$

Thus,

$$\theta_z = \begin{pmatrix} (1_p - zz^*)^{1/2} & 0 \\ 0 & (1_q - z^*z)^{-1/2} \end{pmatrix}.$$

Thus, replace $\overline{\eta}$ by

$$\xi(g) = \xi_s(g) = \overline{\eta(g)} \cdot \det(1_p - zz^*)^s \quad \text{(for Cartan decomposition } g = h_z \cdot k)$$

where $s \in \mathbb{C}$, $\Re(s) \geq 0$. Thus $\xi_{s=0} = \overline{\eta}$. For $\Re(s)$ sufficiently large (positive), $\xi \in L^1(G)$. (This is clear just below.) The integral with the convergence factor inserted becomes

$$T_s f(g) = \int_G f(gh)\, \xi_s(h)\, dh.$$

For s such that $\xi = \xi_s$ is in $L^1(G)$, the integral for $T_s f$ lies in the representation space V generated by f; so for $\Re(s)$ sufficiently large the function $s \longrightarrow T_s f$ is a holomorphic V-valued function of s. Since η was K-conjugation invariant, and since the convergence factor is left and right K-invariant, ξ is K-conjugation invariant for all s. Thus, at least for $\Re(s)$ large, the map $f \longrightarrow T_s f$ maps V to V, commutes with K, and stabilizes K-isotypes in V. The representation V contains only a single copy of the extreme K-type τ, so

[4] The modified integral with complex parameter s is *not* the same as the integral against a section of degenerate principal series on $U(n,n)$ parametrized as usual by s. That is, we do *not* compute the archimedean zeta integral as a function of the *usual* s. Rather, we compute the *value* of the zeta integral at a special point, via analytic continuation.

$f \longrightarrow T_s f$ is a *scalar* map on the copy of τ inside π. Thus, there is a constant Ω_s depending only upon τ (and depending holomorphically upon s) such that

$$T_s f = \Omega_s \cdot f.$$

For $\Re(s)$ sufficiently positive that we can use a different model, we do change the model of the representation to compute Ω_s.

There is *pointwise* (in g) equality

$$T_s f(g) = \int_G f(gh)\, \xi_s(h)\, dh$$

of holomorphic $\mathbb{C}$-valued functions, so the same *pointwise* identity

$$T_s f(g) = \int_G f(gh)\, \xi_s(h)\, dh = \Omega_s \cdot f(g)$$

holds for the analytic continuation. Since the far right-hand side is simply a scalar multiple of f, this ruse gives the desired result even for η not in $L^1(G)$.

To compute the constant Ω_s, it suffices to compute a single pointwise value

$$T_s f(1) = \int_G f(h)\, \xi_s(h)\, dh = \Omega_s \cdot f(1).$$

For $\Re(s)$ sufficiently large such that $\xi_s \in L^1(G)$, we are entitled to take a different model. Take f to be in

$$\tau \otimes \tau^\vee \subset \pi \otimes \pi^\vee \subset L^2(G).$$

Then use the Harish-Chandra decomposition and express f as a finite sum of functions of the special form

$$\begin{aligned} f(h) = f(h_z \cdot k) = f_{u,v}(h_z \cdot k) = f_{u,v}(n_+ \theta_z n_- k) = f_{u,v}(n_+\, \theta_z k\, k^{-1} n_z^- k) \\ = f_{u,v}(\theta_z k) = \langle \tau(\theta_z k) u, v\rangle = \langle \tau(k) u, \tau(\theta_z) v\rangle \end{aligned}$$

where $h = h_z \cdot k$ is the Cartan decomposition, $h_z = n_+ \cdot \theta_z \cdot n_-$ is the Harish-Chandra decomposition, $u, v \in \tau$, and

$$\tau(\theta_z)^* = \tau(\theta_z^*) = \tau(\theta_z).$$

(This uses the holomorphic extension of τ to $K_{\mathbb{C}}$.) Then

$$\Omega_s \cdot f(1) = T_s f(1) = \int_G f(h)\, \xi(h)\, dh = \int_C \int_K f_{u,v}(h_z k)\, \xi(h_z k)\, dk\, d^* z.$$

Recall that ξ is

$$\xi(h_z k) = \overline{\eta}(n_z^+ \theta_z n_z^- k) \cdot \det(1_p - zz^*)^s = \dim \tau \cdot \overline{\mathrm{tr} \tau(\theta_z k)} \cdot \det(1_p - zz^*)^s$$

with the special choice of η. Thus, the integral is

$$Tf(1) = \dim\tau \int_C \int_K \langle \tau(k)u,\ \tau(\theta_z)v\rangle\ \overline{\mathrm{tr}\tau(\theta_z k)}\ \det(1_p - zz^*)^s\, dk\, d^*z.$$

Let $\{x_i\}$ be an orthonormal basis for τ and take the special choice of η as earlier, namely

$$\eta(n_+\theta n_-) = \dim\tau \cdot \mathrm{tr}\tau(\theta) = \dim\tau \cdot \sum_i \langle \tau(\theta)x_i, x_i\rangle.$$

Then, unsurprisingly, part of the integral is computed via Schur's orthogonality relations, namely, [5]

$$\sum_i \int_K \langle \tau(k)u, \tau(\theta_z)v\rangle \overline{\langle \tau(\theta_z k)x_i, x_i\rangle} dk = \sum_i \int_K \langle \tau(k)u, \tau(\theta_z)v\rangle \overline{\langle \tau(k)x_i, \tau(\theta_z)x_i\rangle} dk$$

$$= \sum_i \frac{\langle u, x_i\rangle}{\dim\tau} \langle \tau(\theta_z)x_i, \tau(\theta_z)v\rangle = \sum_i \frac{\langle u, x_i\rangle}{\dim\tau} \langle x_i, \tau(\theta_z^2)v\rangle = \frac{\langle u, \tau(\theta_z^2)v\rangle}{\dim\tau}$$

where K has total measure 1 and use $\tau(\theta_z)^* = \tau(\theta_z^*) = \tau(\theta_x)$. Thus, the factor $\dim\tau$ cancels, as planned, and

$$\Omega_s \cdot f(1) = T_s f(1) = \textstyle\int_C \langle u, \tau(\theta_z^2)v\rangle\ (1_p - zz^*)^s\ d^*z$$

$$= \langle u,\ \left(\textstyle\int_C \tau(\theta_z^2)\ \det(1_p - zz^*)^s\ d^*z\right)\ v\rangle = \langle u,\ Sv\rangle$$

where $S = S_s$ is the endomorphism (anticipated to be a scalar)

$$S = S_s = S_s(\tau) = \int_C \tau(\theta_z^2)\ \det(1_p - zz^*)^s\ d^*z.$$

Here, as above,

$$\theta_z^2 = \begin{pmatrix} 1_p - zz^* & 0 \\ 0 & (1_q - z^*z)^{-1} \end{pmatrix}.$$

The irreducible τ of $K \approx U(p) \times U(q)$ necessarily factors as a tensor product

$$\tau \approx \tau_1 \otimes \tau_2$$

with irreducibles τ_1 of $U(p)$ and τ_2 of $U(q)$. Thus, the endomorphism S of the finite-dimensional vector space τ is

$$S = \int_{D_{p,q}} \tau_1(1_p - zz^*) \otimes \tau_2^{-1}(1_q - z^*z)\ \det(1_p - zz^*)^s\ d^*z \in \mathrm{End}_{\mathbb{C}}(\tau).$$

[5] Since we use the holomorphic extension of τ to $K_{\mathbb{C}}$, there is a minor hazard of a needless false assertion about the application of Schur orthogonality. The explicit argument makes clear that this invocation is legitimate.

We claim that the endomorphism S of the finite-dimensional complex vector space τ is *scalar*. Indeed, the map $z \longrightarrow \alpha z \beta^*$ with $\alpha \in U(p)$, $\beta \in U(q)$ is an automorphism of $D_{p,q}$ that leaves the invariant measure unchanged. Applying this change of variables inside the integral defining S,

$$S = \int_{D_{p,q}} \tau_1(\alpha(1_p - zz^*)\alpha^*) \otimes \tau_2^{-1}(\beta(1_q - z^*z)\beta^*) \det(1_p - zz^*)^s \, d^*z$$

$$= [\tau_1(\alpha) \otimes \tau_2(\beta)] \cdot S \cdot [\tau_1(\alpha) \otimes \tau_2(\beta)]^{-1}.$$

Thus, S commutes with $\tau(k)$ for $k \in K$. By Schur's lemma, since τ is irreducible as a representation of K (apart from its behavior on $K_{\mathbb{C}}$), the endomorphism S of τ is *scalar*, as claimed.

Thus, with scalar S and $f = f_{u,v}$,

$$\Omega_s \cdot f(1) = S \cdot \langle u, v \rangle = S \cdot f(1).$$

Thus, in fact, as an operator on τ,

$$\Omega_s = \int_{D_{p,q}} \tau_1(1_p - zz^*) \otimes \tau_2^{-1}(1_q - z^*z) \det(1_p - zz^*)^s \, d^*z.$$

Let $z = \alpha r \beta$ with $\alpha \in U(p)$, $\beta \in U(q)$, and where r is diagonal (even if rectangular)

$$r = \ p\text{-by-}q \ = \begin{pmatrix} r_1 & & \\ & \ddots & \\ & & r_q \\ 0 & \ldots & 0 \end{pmatrix}$$

with diagonal entries $-1 < r_i < 1$. Let

$$\Delta(r) = \prod_{1 \le i < j \le q} \left(r_i^2 - r_j^2\right)^2.$$

Then, with a constant C determined subsequently, the integral of a function φ on the domain $D_{p,q}$ is [6]

$$\int_{D_{p,q}} \varphi(z) \frac{dz}{\det(1_q - z^*z)^{p+q}} = C \cdot \int_{U(p)\times U(q)} \int_{(-1,1)^q} \varphi(\alpha r \beta) d\alpha d\beta \frac{\Delta(r) dr}{\det(1_q - r^*r)^{p+q}}$$

where $U(p)$ and $U(q)$ have total measure 1 and dr is the product of usual (additive) Haar measures on intervals $(-1, +1)$. Thus,

$$S = C \cdot \int_{U(p)\times U(q)} [\tau_1(\alpha) \otimes \tau_2(\beta)] \circ I \circ [\tau_1(\alpha) \otimes \tau_2(\beta)]^{-1} \, d\alpha \, d\beta$$

[6] This formula can be construed as a variant of Weyl's integration formula. It can be derived directly by considering the exponential map to the symmetric space.

where the inner operator is

$$I = \int_{(-1,1)^q} \tau_1(1_p - rr^*) \otimes \tau_2^{-1}(1_q - r^*r) \frac{\det(1_q - r^*r)^s \, \Delta(r) \, dr}{\det(1_q - r^*r)^{p+q}}.$$

The outer integration is exactly the projection

$$\mathrm{End}_{\mathbb{C}}(\tau) \longrightarrow \mathrm{End}_{U(p)\times U(q)}(\tau)$$

where $\mathrm{End}_{\mathbb{C}}(\tau)$ has the natural $K = U(p) \times U(q)$ structure

$$k \cdot \varphi = \tau(k) \circ \varphi \circ \tau(k)^{-1}$$

for $k \in K$ and $\varphi \in \mathrm{End}_{\mathbb{C}}(\tau)$.

Give $\mathrm{End}_{\mathbb{C}}(\tau)$ an E-rational structure compatible with whatever E-rational structure

$$\mathfrak{k}_E = \mathfrak{gl}(p) \otimes \mathfrak{gl}(q),$$

we have on the complexified Lie algebra $\mathfrak{k}_{\mathbb{C}} = \mathfrak{gl}(p,\mathbb{C}) \otimes \mathfrak{gl}(q,\mathbb{C})$ of $K = U(p) \times U(q)$ under the action of $\mathfrak{k}_E$. We will directly compute that the inner integral for S is in

$$\left(\frac{2^s\,\Gamma(s)^2}{\Gamma(2s)}\right)^q \cdot E(s) \qquad (E(s) = \text{rational functions in } s\text{, coefficients in } E).$$

That is, up to the indicated leading factor, the inner integral is a rational function of s with coefficients in E. Then we show that the projection P to K-invariants is an element in the center $Z(\mathfrak{k}_E)$ of the enveloping algebra $U = U(\mathfrak{k}_E)$ of $\mathfrak{k}_E$. Thus, apart from the normalizing constant C, up to the leading factor the whole integral for S is again in $E(s)$. Then we determine the constant C by a different computation of S with a special choice of $\tau = \tau_1 \otimes \tau_2$.

Any finite-dimensional representation of $U(n)$ gives a representation of the complexified Lie algebra $\mathfrak{gl}(n,\mathbb{C})$ of $U(n)$ with a highest weight. In the context in which these integrals arise, one often does not have a $\mathbb{Q}$-rational structure, but only over a CM-field or totally real subfield of such. Consider the *E-rational* Lie algebra $\mathfrak{gl}(n,E)$. Following Dixmier,[7] construct the universal enveloping algebra $U(\mathfrak{gl}(n,E))$ over E, and the Poincaré–Birkhoff–Witt theorem holds. Let $\mathfrak{n}$ be the strictly upper-triangular subalgebra, and $\mathfrak{a}$ the diagonal subalgebra. For an E-rationally-valued character $\lambda : \mathfrak{a} \longrightarrow E$ of $\mathfrak{a}$ (following Dixmier) we have the Verma module over E

[7] J. Dixmier, *Enveloping Algebras*, (English translation 1995, A.M.S.) North-Holland, 1977, shows that essentially all the usual algebraic constructions depend only upon the underlying field being of characteristic 0. Specifically, behavior of the enveloping algebra, Verma modules, and expression of finite-dimensional irreducibles as quotients of Verma modules proceeds over arbitrary fields of characteristic 0, with the obvious trivial modifications, in the fashion usually carried out over $\mathbb{C}$.

$$M_\lambda = U / \left(U \cdot \mathfrak{n} \oplus \sum_\alpha U \cdot (\alpha - (\lambda - \rho)\alpha) \right)$$

as the quotient of U by the left ideal generated by $\mathfrak{n}$ and the differences $\alpha - (\lambda - \rho)\alpha$ for $\alpha \in \mathfrak{a}$, with ρ the usual half-sum of positive roots (*positive* roots being the roots of $\mathfrak{a}$ in $\mathfrak{n}$). As over $\mathbb{C}$, there is a unique maximal proper submodule N_λ, namely, the sum of all submodules not containing the $\lambda - \rho$ weight space. As over $\mathbb{C}$, the E-irreducible quotient M_λ / N_λ is finite-dimensional for λ *integral* and *dominant.* Further, this construction commutes with extension of scalars to $\mathbb{C}$, so $(M_\lambda / N_\lambda) \otimes \mathbb{C}$ is still irreducible.

Since the highest weights $\lambda - \rho$ for finite-dimensional irreducibles are *integral* (and dominant), they certainly take E-rational values on the E-rational diagonal subalgebra $\mathfrak{a}_E$. Therefore, any finite-dimensional irreducible complex representation τ of $U(n)$ has an E-rational structure

$$\tau = (M_\lambda / N_\lambda) \otimes_E \mathbb{C}$$

for some λ, with an *E-rational* Verma module M_λ as above. Let τ_E be such an E-rational form of τ, and $\tau_E^\vee$ an E-rational form of its dual. An irreducible τ of $U(p) \times U(q)$ factors as $\tau \approx \tau_1 \otimes \tau_2$ with irreducibles τ_1 of $U(p)$ and τ_2 of $U(q)$, so we can choose E-rational structures $\tau_{i,E}$ on τ_i and put

$$\tau_E = \tau_{1,E} \otimes_E \tau_{2,E}.$$

We have the usual natural isomorphism

$$\mathrm{End}_E(\tau_E) \approx \tau_E \otimes \tau_E^\vee.$$

Let P be the projection to the trivial $K = U(p) \times U(q)$ subrepresentation inside $\mathrm{End}_E(\tau_E) \otimes_E \mathbb{C}$. (The *absolute* irreducibility and Schur's lemma together assure that this subspace is exactly one-dimensional, and consists of scalar operators.) We want to show that

$$P\left(\mathrm{End}_E(\tau_E)\right) \subset E \cdot \mathrm{id}_\tau \subset \mathrm{End}_\mathbb{C}(\tau_\mathbb{C})$$

that is, that E-rational $\mathbb{C}$-endomorphisms project to E-rational (scalar) K-endomorphisms.

Following Dixmier, the Harish-Chandra isomorphism and its proof hold for the E-rational Lie algebra $\mathfrak{k}_E = \mathfrak{gl}(p, E) \otimes \mathfrak{gl}(q, E)$. In particular, the center Z_E of $U(\mathfrak{k}_E)$ distinguishes finite-dimensional irreducible $\mathfrak{k}_E$-modules with distinct E-rational highest weights. That is, given finite-dimensional irreducibles V and V' with E-rational highest weights $\lambda \neq \lambda'$, there is $z \in Z_E$ such that $z(\lambda) = 1$ and $z(\lambda') = 0$. Then, with Λ the *finite* collection of λ's indexing the irreducibles in $\mathrm{End}_\mathbb{C}(\tau) = \mathrm{End}_E(\tau_E) \otimes_E \mathbb{C}$,

$$P = \prod_{\lambda \in \Lambda} z_\lambda$$

projects endomorphisms to the K-invariant subspace. Thus, projection to K-endomorphisms preserves E-rationality.

The diagonal subgroups in $GL(p,\mathbb{C})$ and $GL(q,\mathbb{C})$ act on the weight spaces in τ. The inner integral in the description of S preserves each weight space and acts on it by a scalar. Noting the identity

$$(t^2-u^2) = (t^2-1)-(u^2-1)$$

one can see that each such scalar is expressible as a sum with E-rational coefficients of products of integrals of the form (with $n \in \mathbb{Z}$)

$$\int_{-1}^{1} (1-t^2)^n \, \frac{(1-t^2)^s\,dt}{(1-t^2)^{p+q}} = \int_0^1 [4t(1-t)]^n \, \frac{2\,[4t(1-t)]^s\,dt}{[4t(1-t)]^{p+q}}$$

$$= 2^{2n+1-p-q+s} \int_0^1 t^{n-p-q}\,(1-t)^{n-p-q+s}\,dt$$

$$= 2^{2n+1-p-q+s}\,\frac{\Gamma(n-p-q+1+s)\,\Gamma(n-p-q+1+s)}{\Gamma(2n-2p-2q+2+2s)} \in \frac{2^s\Gamma(s)^2}{\Gamma(2s)}\cdot E(s)$$

(replacing t by $2t-1$ in the first expression). Thus, the inner integral acts by scalars on all weight spaces, and these scalars (up to the leading factor) are in $E(s)$. Thus, when $s=0$, this endomorphism is in $\mathrm{End}_E(\tau_E)$. We have shown that projection to $(U(p)\times U(q))$-endomorphisms preserves E-rationality, so

$$S_{s=0} = C\cdot(E\text{-rational scalar endomorphism of }\tau).$$

To determine the normalization constant C, it suffices to compute the endomorphism

$$S = S(\tau) = \int_{D_{p,q}} \tau_1(1_p - zz^*)\otimes\tau_2^{-1}(1_q - z^*z)\,d^*z$$

for any particular choice of $\tau=\tau_1\otimes\tau_2$. The following is the simplest version of the fuller qualitative computation done in the following section.

Let

$$\tau_1(x) = (\det x)^m \quad (\text{on } GL(p,\mathbb{C})) \qquad \text{and} \qquad \tau_2(y) = (\det y)^{-n} \quad (\text{on } GL(q,\mathbb{C}))$$

with m and n sufficiently large to assure that we are in the L^1 range. For positive integer ℓ, define the cone

$$C_\ell = \{\text{positive-definite Hermitian } \ell\text{-by-}\ell \text{ complex matrices}\}$$

and associated gamma function

$$\Gamma_\ell(s) = \int_{C_\ell} e^{-\mathrm{tr} x}\,(\det x)^s \frac{dx}{(\det x)^\ell} \qquad\qquad (\text{for } \Re(s) > \ell-1),$$

where dx denotes the product of the usual measures on $\mathbb{R}$ for the diagonal components of x, and the usual measure on $\mathbb{C} \approx \mathbb{R}^2$ for the off-diagonal components. It is a classical fact [8] and is straightforward to determine directly that

$$\Gamma_\ell(s) = \pi^{\ell(\ell-1)/2} \prod_{i=1}^{\ell} \Gamma(s-i+1).$$

Let $x^{1/2}$ and $y^{1/2}$ be the unique positive-definite Hermitian square roots of $x \in C_p$ and $y \in C_q$. Then (with arguments to Γ_p and Γ_q that are intelligible only with some hindsight)

$$\det(x)^{m+n} \det(y)^{m+n} \cdot S$$
$$= (\det x)^n (\det y)^m \left[\det(x^{1/2})^m \det(y^{1/2})^n\right] \cdot S \cdot \left[\det(x^{1/2})^m \cdot \det(y^{1/2})^n\right].$$

Multiply both sides by $e^{-\mathrm{tr}x-\mathrm{tr}y}$ and integrate over $C_p \times C_q$ against the invariant measures $(\det x)^{-p}dx$ and $(\det y)^{-q}dy$ to obtain

$$\Gamma_p(m+n)\, \Gamma_q(m+n) \cdot S =$$
$$\int_{C_p\times C_q\times D_{p,q}} e^{-\mathrm{tr}x-\mathrm{tr}y} \det(x-\sqrt{x}zz^*\sqrt{x})^n \det(y-\sqrt{y}z^*z\sqrt{y})^m \frac{(\det x)^{n-p}(\det y)^{m-q}dxdydz}{\det(1_p-zz^*)^{p+q}}.$$

Replacing $z \in D_{p,q}$ by $x^{-1/2}zy^{-1/2}$ converts the integral over $C_p \times C_q \times D_{p,q}$ into an integral over

$$Z = \begin{pmatrix} x & z \\ z^* & y \end{pmatrix} \in C_{p+q}.$$

Replacing z by $x^{-1/2}zy^{-1/2}$ replaces the additive Haar measure measure dz by

$$(\det x)^{-q} (\det y)^{-p}\, dz.$$

The normalizing factor $\det(1_p-zz^*)^{-(p+q)}$ associated with dz can be rewritten as

$$\det(1_p - zz^*)^{-(p+q)} = \det(1_p - zz^*)^{-q} \det(1_q - z^*z)^{-p}$$

and then under the replacement of z by $x^{-1/2}zy^{-1/2}$ becomes

$$\det(1_p - x^{-1/2}zy^{-1}z^*x^{-1/2})^{-q} \det(1_q - y^{-1/2}z^*x^{-1}zy^{-1/2})^{-p}$$
$$= (\det x)^q \det(x - zy^{-1}z^*)^{-q} (\det y)^p \det(y - zx^{-1}z^*)^{-p}.$$

[8] Gamma functions attached to cones appeared in C.L. Siegel *Über die analytische Theorie der quadratischen Formen*, Ann. Math. **36** (1935), pp. 527–606, C.L. Siegel *Über die Zetafunktionene indefinier quadratischer Former*, Math. Z. **43** (1938), pp. 682–708, H. Maaß *Siegel's modular forms and Dirichlet series*, SLN 216, Berlin, 1971. See also the application in G. Shimura *Confluent hypergeometric functions on tube domains*, Math. Ann. **260** (1982), pp. 269–302. In any case, the indicated identity follows readily by an induction, and is elementary.

Putting this together, the integral becomes

$$\int_{C_{p+q}} e^{-\mathrm{tr} Z} \det(x - zy^{-1}z^*)^m \det(y - z^*x^{-1}z)^n \frac{(\det x)^{n-p} (\det y)^{m-q} \, dx \, dy \, dz}{\det(x - zy^{-1}z^*)^q \det(y - z^*x^{-1}z)^p}.$$

The identity

$$\begin{pmatrix} x & z \\ z^* & y \end{pmatrix} = \begin{pmatrix} 1 & zy^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x - zy^{-1}z^* & 0 \\ 0 & y \end{pmatrix} \begin{pmatrix} 1 & 0 \\ y^{-1}z^* & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ z^*x^{-1} & 1 \end{pmatrix} \begin{pmatrix} x & 0 \\ 0 & y - z^*x^{-1}z \end{pmatrix} \begin{pmatrix} 1 & x^{-1}z \\ 0 & 1 \end{pmatrix}$$

shows that

$$\det Z = \det(x - zy^{-1}z^*) \cdot \det y = \det(y - z^*x^{-1}z) \cdot \det x.$$

Thus, altogether,

$$\Gamma_p(m+n)\Gamma_q(m+n) \cdot S = \int_{C_{p+q}} e^{-\mathrm{tr} Z} (\det Z)^{m+n} \frac{dZ}{(\det Z)^{p+q}} = \Gamma_{p+q}(m+n).$$

Thus, for this particular choice of τ,

$$S = \frac{\Gamma_{p+q}(m+n)}{\Gamma_p(m+n-p)\,\Gamma_q(m+n-q)}$$

$$= \frac{\prod_{i=0}^{p+q-1} \Gamma(m+n-i+1)}{\prod_{i=0}^{p-1} \Gamma(m+n-i+1) \cdot \prod_{i=0}^{q-1} \Gamma(m+n-i+1)} \cdot \frac{\pi^{(p+q)(p+q-1)/2}}{\pi^{p(p-1)/2} \cdot \pi^{q(q-1)/2}}.$$

The net exponent of π is

$$(p+q)(p+q-1)/2 - p(p-1)/2 - q(q-1)/2 = pq$$

as anticipated. Thus, *indirectly*, we have shown that

$$C = \pi^{pq} \cdot (E\text{-rational number})$$

where necessarily the E-rational number is independent of m, n, being a normalization of a measure. Then, for *arbitrary* τ, evaluating at $s = 0$,

$$S = \pi^{pq} \cdot (E\text{-rational scalar endomorphism of } \tau)$$

This proves the *qualitative* assertion formulated above. □

3 Quantitative theorem

The simple case

$$\tau(k_1 \times k_2) = (\det k_1)^m \, (\det k_2)^{-n} \ \ (m \geq p, \, n \geq q, \, k_1 \in U(p), \, k_2 \in U(q))$$

used in the last section to determine a normalization gives an explicit form of the dependence upon the representation τ, namely that (up to a uniform non-zero E-rational number) in that special case

$$Tf = \pi^{pq} \cdot \frac{\prod_{i=0}^{p+q-1} \Gamma(m+n-i)}{\prod_{i=0}^{p-1} \Gamma(m+n-p-i) \cdot \prod_{i=0}^{q-1} \Gamma(m+n-q-i)} \cdot f.$$

This computation can be extended, as we do in this section, at least to the case that only *one* of the two τ_i is one-dimensional.

As discussed in the introductory section, any function φ in the copy of $\tau \otimes \tau^\vee$ inside $\pi_\tau \otimes \pi_\tau^\vee$ is completely determined by its values on $K_\mathbb{C}$, and on $K_\mathbb{C}$ the function φ is holomorphic and lies in the (holomorphic extension of the) unique copy of $\tau \otimes \tau^\vee$ inside $L^2(K)$. The latter consists of (holomorphic extensions of) (matrix) coefficient functions

$$c_{u,v}(\theta) = \langle \tau(\theta)u, v \rangle \qquad\qquad (u, v \in \tau)$$

where $\langle \, , \rangle$ is a fixed K-invariant Hermitian inner product on τ. We can assume that for g in the complexification $K_\mathbb{C}$

$$\tau(g^*) = \tau(g)^*$$

where $g \longrightarrow g^*$ is the involution on $K_\mathbb{C}$ which fixes the real points K of $K_\mathbb{C}$. Thus, we may take f of the form (in the $N_+ K_\mathbb{C} N_-$ coordinates)

$$f(g) = f_{u,v}(g) = f_{u,v}(n_+ \theta n_-) = c_{u,v}(\theta).$$

As recalled above, for extreme K-type τ with sufficiently high extreme weight the universal $(\mathfrak{g}, K)$-module generated by a vector v_τ of K-type τ and annihilated by $\mathfrak{p}_-$ is irreducible. Thus, the annihilation of η by $\mathfrak{p}_-$ and the specification of K-type of η imply that η generates the holomorphic discrete series $\tau \otimes \tau^\vee$ and that η is a finite sum

$$\eta_{\mu,\nu}(n_+ \theta n_-) = c_{\mu,\nu}(\theta) = \langle \tau(\theta)\mu, \nu \rangle$$

with $\mu, \nu \in \tau$. Use the same normalization of measure on G as earlier.

Let τ_1 have extreme weights $(\kappa_1, \ldots, \kappa_p)$, and let

$$\kappa_{p+1} = \cdots = \kappa_{p+q} = \kappa_p$$

Let τ_2 have extreme weights $(\lambda_{p+1}, \ldots, \lambda_{p+q})$ (note the funny indexing!), and let

$$\lambda_1 = \lambda_2 = \cdots = \lambda_p = \lambda_{p+1}.$$

Note also that in practice these λ_i will most often be negative integers. For example, the scalar example of the previous section is recovered by taking $\lambda_i = -n$, with n as in the previous section.

A similar computation, as in the special case of the previous section, will yield:

Theorem 3.1. *For $\tau = \tau_1 \otimes \tau_2$ with either τ_1 or τ_2 one-dimensional,*

$$\begin{aligned}(Tf)(1) &= \int_G f_{u,v}(h)\,\overline{\eta_{\mu,\nu}(h)}\,dh \\ &= \pi^{pq} \cdot \langle u,\mu\rangle \cdot \langle v,\nu\rangle \cdot \frac{\prod_{i=1}^{p+q} \Gamma(\kappa_i-(p+q-i)-\lambda_i)}{\prod_{i=1}^{p} \Gamma(\kappa_i-(p-i)-\lambda_p)\ \prod_{i=1}^{p+q} \Gamma(\kappa_p-(q-i)-\lambda_i)}.\end{aligned}$$

Proof. As in the proof of the previous theorem, Theorem 2.1, use a Cartan decomposition $h = h_z k$ and the Harish-Chandra decomposition $h_z = n_z^+ \theta_z n_z^-$

$$\begin{aligned}h_z &= \begin{pmatrix} (1_p - zz^*)^{-1/2} & z(1_q - z^*z)^{-1/2} \\ (1_q - z^*z)^{-1/2}z^* & (1_q - z^*z)^{-1/2} \end{pmatrix} \\ &= \begin{pmatrix} 1_p & z \\ 0 & 1_q \end{pmatrix} \begin{pmatrix} (1_p - zz^*)^{1/2} & 0 \\ 0 & (1_q - z^*z)^{-1/2} \end{pmatrix} \begin{pmatrix} 1_q & 0 \\ z^* & 1_q \end{pmatrix}.\end{aligned}$$

Thus

$$\theta_z = \begin{pmatrix} (1_p - zz^*)^{1/2} & 0 \\ 0 & (1_q - z^*z)^{-1/2} \end{pmatrix}$$

and the integral is

$$(Tf)(1) = \int_G f_{u,v}(h)\,\overline{\eta_{\mu,\nu}(h)}\,dh = \int_C \int_K f_{u,v}(h_z k)\,\overline{\eta_{\mu,\nu}(h_z k)}\,dk\,dz^*.$$

The special form of $f_{u,v}$ gives

$$f_{u,v}(h_z k) = f_{u,v}(n_z^+ \theta_z n_z^- k) = f_{u,v}(\theta_z k \cdot k^{-1} n_z^- k) = f_{u,v}(\theta_z k) = \langle \tau(\theta_z k)u, v\rangle$$

since $\theta_z k \in K_{\mathbb{C}}$. Similarly for $\eta_{\mu,\nu}$. As in the previous proof, we insert a convergence factor $\det(1_p - zz^*)^s$, so the integral is

$$\begin{aligned}&\int_C \int_K \langle \tau(\theta_z k)u, v\rangle\,\overline{\langle \tau(\theta_z k)\mu, \nu\rangle}\,\det(1_p - zz^*)^s\,dk\,d^*z \\ &= \int_C \int_K \langle \tau(k)u, \tau(\theta_z)^* v\rangle\,\overline{\langle \tau(k)\mu, \tau(\theta_z)^*\nu\rangle}\,\det(1_p - zz^*)^s\,dk\,d^*z.\end{aligned}$$

The Schur inner product relations compute the integral over K, leaving

$$(Tf)(1) = \frac{\langle u,\mu\rangle}{\dim \tau} \cdot \int_C \langle \tau(\theta_z)^*\nu, \tau(\theta_z)^* v\rangle\,\det(1_p - zz^*)^s\,d^*z.$$

Rearrange this slightly, to

$$\begin{aligned}&\frac{\langle u,\mu\rangle}{\dim\tau}\cdot\int_C \langle \nu, \tau(\theta_z)\tau(\theta_z)^* v\rangle\ \det(1_p - zz^*)^s\, d^*z\\ &= \frac{\langle u,\mu\rangle}{\dim\tau}\cdot\int_C \langle \nu, \tau(\theta_z^2)\, v\rangle\ \det(1_p - zz^*)^s\, d^*z\\ &= \frac{\langle u,\mu\rangle}{\dim\tau}\cdot\langle \nu, \left(\int_C \tau(\theta_z^2)\ \det(1_p - zz^*)^s\, d^*z\right)\, v\rangle\end{aligned}$$

since $\tau(g^*) = \tau(g)^*$ for g in $K_{\mathbb{C}}$, and since $\theta_z^* = \theta_z$. Thus, we need to compute the endomorphism

$$S = \int_C \tau(\theta_z^2)\ \det(1_p - zz^*)^s\, d^*z.$$

As above,

$$\theta_z = \begin{pmatrix} (1_p - zz^*)^{1/2} & 0 \\ 0 & (1_q - z^*z)^{-1/2} \end{pmatrix}$$

so

$$\theta_z^2 = \begin{pmatrix} 1_p - zz^* & 0 \\ 0 & (1_q - z^*z)^{-1} \end{pmatrix}.$$

The irreducible τ of $K \approx U(p) \times U(q)$ necessarily factors as an external tensor product

$$\tau \approx \tau_1 \otimes \tau_2$$

with irreducibles τ_1 of $U(p)$ and τ_2 of $U(q)$. Thus, the endomorphism S of the finite-dimensional vector space τ is

$$S = S_s = \int_{D_{p,q}} \tau_1(1_p - zz^*) \otimes \tau_2^{-1}(1_q - z^*z)\ \det(1_p - zz^*)^s\, d^*z \in \mathrm{End}_{\mathbb{C}}(\tau).$$

As noted in the proof of the qualitative result above, the endomorphism S of the finite-dimensional complex vector space τ is *scalar*, since a change of variables in the defining integral shows that it commutes with $\tau(k)$ for all $k \in K$, and we invoke Schur's lemma.

Even though we seem forced eventually to make the restrictive hypothesis that τ_2 (or τ_1) is *scalar*, we set up the general form of the computation. Again, let τ_1 have extreme weight $(\kappa_1, \ldots, \kappa_p)$ and τ_2 have extreme weight $(\lambda_{p+1}, \ldots, \lambda_{p+q})$, where our convention is that the extreme weight vectors $v_1 \in \tau_1$ and $v_2 \in \tau_2$ satisfy

$$\tau_1 \begin{pmatrix} m_1 & & 0 \\ & \ddots & \\ * & & m_p \end{pmatrix} v_1 = m_1^{\kappa_1} \cdots m_p^{\kappa_p} \cdot v_1,$$

$$\tau_2 \begin{pmatrix} m_1 & & 0 \\ & \ddots & \\ * & & m_q \end{pmatrix} v_2 = m_1^{\lambda_{p+1}} \cdots m_q^{\lambda_{p+q}} \cdot v_2.$$

We introduce a notion of gamma functions somewhat more general than that presented earlier (still falling into a family treated long ago by Siegel and others). Again, for positive integer n, let

$$C_n = \{\text{positive-definite hermitian } n\text{-by-}n \text{ complex matrices}\}.$$

For an irreducible representation σ of $U(n)$, extend σ to a holomorphic representation of $GL(n,\mathbb{C})$, and for complex s define

$$\Gamma_n(\sigma,s) = \int_{C_n} e^{-\mathrm{tr} x}\,\sigma(x)\,(\det x)^s \frac{dx}{(\det x)^n} \in \mathrm{End}_{\mathbb{C}}(\sigma)$$

where dx denotes the product of the usual measures on $\mathbb{R}$ for the diagonal components of x, and the usual measure on $\mathbb{C}$ for the off-diagonal components.

Proposition 3.2. *The endomorphism-valued function $\Gamma_n(\sigma,s)$ is scalar, and can be evaluated in terms of the extreme weight of σ, namely, with extreme weight*

$$(\sigma_1,\ldots,\sigma_n) \quad \textit{with } \sigma_1 \le \cdots \le \sigma_n.$$

We thus have

$$\Gamma_n(\sigma,s) = \pi^{n(n-1)/2} \prod_{i=1}^{n} \Gamma(\sigma_i - (n-i) + s).$$

Proof. For $\alpha \in U(n)$, replacing x by $\alpha x \alpha^*$ in the integral for $\Gamma_n(\sigma,s)$ yields

$$\Gamma_n(\sigma,s) = \int_{C_n} e^{-\mathrm{tr}(\alpha x \alpha^*)}\,\sigma(\alpha x \alpha^*)\,(\det x)^s\, d^*x$$

$$= \sigma(\alpha)\cdot \int_{C_n} e^{-\mathrm{tr}(\alpha^* \alpha x)}\,\sigma(x)\,(\det x)^s\, d^*x \cdot \sigma(\alpha)^{-1} = \sigma(\alpha)\cdot \Gamma_n(\sigma,s)\cdot\sigma(\alpha)^{-1}.$$

Thus, $\Gamma_n(\sigma,s)$ commutes with $\sigma(\alpha)$ for all $\alpha \in U(n)$. By Schur's lemma, $\Gamma_n(\sigma,s)$ is scalar. Replacing x by $y^{1/2}xy^{1/2}$ and using the fact that $\Gamma_n(\sigma,s)$ is scalar gives the second integral formula.

To compute the scalar, it suffices to compute the effect of the operator on an extreme weight vector for σ. Let P be the group of lower-triangular matrices in $GL(n,\mathbb{C})$ with positive real diagonal entries. This subgroup is still transitive on the cone C_n with the action $g(x) = g^*xg$, so

$$\Gamma_n(\sigma,s) = 2^n \cdot \int_P e^{-\mathrm{tr} p^* p}\,\sigma(p^*p)\,(\det p^*p)^s\, dp$$

with suitable right Haar measure dp. Let $\langle\ ,\ \rangle$ be a K-invariant inner product on σ, which we may suppose satisfies

$$\langle \sigma(g)u, w\rangle = \langle u, \sigma(g^*)w\rangle$$

for all $g \in GL(n, \mathbb{C})$: that is, the Hilbert space adjoint $\sigma(g)^*$ of $\sigma(g)$ is $\sigma(g^*)$, where g^* is conjugate transpose. Then with an extreme weight vector v for σ

$$\begin{aligned}\langle \Gamma_n(\sigma)v, v\rangle &= \int_{C_n} e^{-\mathrm{tr} p^* p} \langle \sigma(p)^* \sigma(p) v, v\rangle \, dp = \int_{C_n} e^{-\mathrm{tr} p^* p} \langle \sigma(p) v, \sigma(p) v\rangle \, dp \\ &= \int_{C_n} e^{-\mathrm{tr} p^* p} \, |m_1|^{2\sigma_1} \cdots |m_n|^{2\sigma_n} \, dp \cdot \langle v, v\rangle\end{aligned}$$

where

$$p = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ u_{ij} & & 1 \end{pmatrix} \begin{pmatrix} m_1 & & 0 \\ & \ddots & \\ 0 & & m_n \end{pmatrix}.$$

The latter integral is

$$\int_{m_i>0, u_{ij}\in\mathbb{C}} e^{-\left[m_1^2(1+|u_{21}|^2+\cdots+|u_{n1}|^2)+m_2^2(1+|u_{32}|^2+\cdots+|u_{n2}|^2)+\cdots+m_n^2\right]} m_1^{2(\sigma_1+s)} \cdots m_n^{2(\sigma_n+s)} dp.$$

Replacing each u_{ij} by $u_{ij}\sqrt{\pi}/m_j$ changes the measure by $\prod_{i>j} \pi/m_j^2$ (since $u_{ij} \in \mathbb{C}$), and each integral over u_{ij} becomes (with usual Haar measure on $\mathbb{C} \approx \mathbb{R}^2$)

$$\int_{\mathbb{C}} e^{-\pi|u|^2} \, du = 1.$$

Thus, the integral is

$$\pi^{\frac{n(n-1)}{2}} \int_{m_1>0,\ldots,m_n>0} e^{-(m_1^2+\cdots+m_n^2)} \, m_1^{2(\sigma_1-(n-1)+s)} \, m_2^{2(\sigma_2-(n-2)+s)} \cdots m_n^{2(\sigma_n+s)} \frac{dm_1}{m_1} \cdots \frac{dm_n}{m_n}$$

$$= \pi^{n(n-1)/2} \cdot 2^{-n} \, \Gamma(\sigma_1-(n-1)+s)\Gamma(\sigma_2-(n-2)+s) \cdots \Gamma(\sigma_{n-1}-1+s)\Gamma(\sigma_n+s)$$

after replacing each m_i by $\sqrt{m_i}$. With respect to the coordinates u_{ij} and m_i above, versus the coordinate $x \in C_n$, the Jacobian determinant of the map $p \longrightarrow p^* p$ is 2^n; so the powers of 2 go away. □

Similarly,

Proposition 3.3. *The operator-valued gamma function defined by*

$$\Gamma_n(\sigma^*, s) = \Gamma_n(\sigma^{*,-1}, s) = \int_{C_n} e^{-\mathrm{tr} x} \, \sigma^{-1}(x) \, (\det x)^s \frac{dx}{(\det x)^n}$$

where σ has extreme weights σ_i, is scalar, given by

$$\Gamma_n(\sigma^*, s) = \pi^{n(n-1)/2} \prod_{i=1}^{n} \Gamma(-\sigma_i - (n-i) + s).$$

Proof. This is a trivial variation on the previous proof. □

Now evaluate S. Since S is scalar, using the unique positive-definite Hermitian square roots of $x \in C_p$ and $y \in C_q$,

$$\tau_1(x) \otimes \tau_2^{-1}(y) \cdot S = \left[\tau_1(x^{1/2}) \otimes \tau_2^{-1}(y^{1/2})\right] \cdot \left[\tau_1(x^{1/2}) \otimes \tau_2^{-1}(y^{1/2})\right] \cdot S$$

$$= \left[\tau_1(x^{1/2}) \otimes \tau_2^{-1}(y^{1/2})\right] \cdot S \cdot \left[\tau_1(x^{1/2}) \otimes \tau_2^{-1}(y^{1/2})\right].$$

Multiply both sides by

$$e^{-\mathrm{tr} x - \mathrm{tr} y} \, (\det x)^{s_1+s} \, (\det y)^{s_2+s}$$

with s_1 and s_2 to be determined later, and integrate over $C_p \times C_q$ against the invariant measure to obtain (keeping in mind that $\tau_2(h^*) = \tau_2(h)^*$)

$$\Gamma_p(\tau_1, s_1 + s) \otimes \Gamma_q(\tau_2^{*,-1}, s_2 + s) \cdot S$$

$$= \int_{C_p \times C_q \times D_{p,q}} e^{-\mathrm{tr} x - \mathrm{tr} y} \tau_1(x - x^{1/2} z z^* x^{1/2}) \otimes \tau_2^{-1}(y - y^{1/2} z^* z y^{1/2})$$

$$\times \frac{dz}{\det(1_p - zz^*)^{p+q-s}} \frac{\det x^{s_1+s} dx}{\det x^p} \frac{\det y^{s_2+s} dy}{\det y^q}.$$

Observe that we must take the transpose-conjugate of τ_2^{-1} in order to have a (anti-holomorphic) representation in the gamma function, though in fact we have

$$\tau_2^*(y) = \tau_2(y^*) = \tau_2(y) \qquad (\text{for } y = y^*).$$

Replacing $z \in D_{p,q}$ by $x^{-1/2} z y^{-1/2}$ converts the integral over $C_p \times C_q \times D_{p,q}$ into an integral over

$$Z = \begin{pmatrix} x & z \\ z^* & y \end{pmatrix} \in C_{p+q}.$$

A change of measure by $(\det x)^{-q} (\det y)^{-p}$ comes out. (The exponents are *not* divided by 2, despite the square roots, since the z variable is *complex*, and each complex coordinate has two real coordinates.) In fact, we want to break the $\det(1_p - zz^*)^{p+q-s}$ into two pieces,

$$\det(1_p - zz^*)^{p+q-s} = \det(1_p - zz^*)^{p-\frac{s}{2}} \cdot \det(1_q - z^* z)^{q-\frac{s}{2}}.$$

We also use the identity

$$\begin{pmatrix} x & z \\ z^* & y \end{pmatrix} = \begin{pmatrix} 1 & zy^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x - zy^{-1}z^* & 0 \\ 0 & y \end{pmatrix} \begin{pmatrix} 1 & 0 \\ y^{-1}z^* & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ z^* x^{-1} & 1 \end{pmatrix} \begin{pmatrix} x & 0 \\ 0 & y - z^* x^{-1} z \end{pmatrix} \begin{pmatrix} 1 & x^{-1} z \\ 0 & 1 \end{pmatrix}$$

which implies that

$$\det Z = \det(x - zy^{-1}z^*) \cdot \det y = \det(y - z^*x^{-1}z) \cdot \det x.$$

Thus,

$$\det(1_p - zz^*)^{p+q-s}(\det x)^{p-s_1-s}(\det y)^{q-s_2-s} = (\det Z)^{p+q-s}(\det x)^{-s_1}(\det y)^{-s_2}$$

and the integral becomes

$$\Gamma_p(\tau_1, s_1 + s) \otimes \Gamma_q(\tau_2^{*,-1}, s_2 + s) \cdot S$$

$$= \int_{C_{p+q}} e^{-\mathrm{tr} Z}\, \tau_1(x - zy^{-1}z^*) \otimes \tau_2^{-1}(y - z^*x^{-1}z)(\det x)^{s_1}\, (\det y)^{s_2}\, \frac{(\det Z)^s\, dZ}{(\det Z)^{p+q}},$$

since the left-hand side of the latter equality is a product of scalar operators, the right-hand side is scalar.

Let $\tilde{\tau}_1$ be the irreducible representation of $GL(p+q, \mathbb{C})$ with extreme weight vector $\tilde{v}_1$ with weight

$$\tilde{\tau}_1 \begin{pmatrix} t_1 & & 0 \\ & \ddots & \\ * & & t_{p+q} \end{pmatrix} \tilde{v}_1 = t_1^{\kappa_1} \cdots t_{p+q}^{\kappa_{p+q}} \cdot \tilde{v}_1$$

where we take

$$\kappa_{p+1} = \kappa_{p+2} = \cdots = \kappa_{p+q} = \kappa_p.$$

The restriction of $\tilde{\tau}_1$ to

$$\{\begin{pmatrix} A & 0 \\ 0 & 1_q \end{pmatrix} : A \in GL(p, \mathbb{C})\} \approx GL(p, \mathbb{C})$$

has (among its several extreme weight vectors) the vector $\tilde{v}_1$ as extreme weight vector with

$$\tilde{\tau}_1 \begin{pmatrix} A & 0 \\ 0 & 1_q \end{pmatrix} \cdot \tilde{v}_1 = t_1^{\kappa_1} \cdots t_p^{\kappa_p} \cdot \tilde{v}_1$$

for lower-triangular A with diagonal entries t_i. Thus, a copy of τ_1 containing $\tilde{v}_1$ lies inside $\tilde{\tau}_1$. Then

$$\langle \tilde{\tau}_1 \begin{pmatrix} x & z \\ z^* & y \end{pmatrix} \tilde{v}_1, \tilde{v}_1 \rangle = \langle \tilde{\tau}_1 \left[\begin{pmatrix} 1 & zy^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x - zy^{-1}z^* & 0 \\ 0 & y \end{pmatrix} \begin{pmatrix} 1 & 0 \\ y^{-1}z^* & 1 \end{pmatrix} \right] \tilde{v}_1, \tilde{v}_1 \rangle$$

$$= \langle \tilde{\tau}_1 \left[\begin{pmatrix} x - zy^{-1}z^* & 0 \\ 0 & y \end{pmatrix} \begin{pmatrix} 1 & 0 \\ y^{-1}z^* & 1 \end{pmatrix} \right] \tilde{v}_1,\ \tilde{\tau}_1 \begin{pmatrix} 1 & y^{-1}z^* \\ 0 & 1 \end{pmatrix} \tilde{v}_1 \rangle$$

$$= \langle \tilde{\tau}_1 \begin{pmatrix} x - zy^{-1}z^* & 0 \\ 0 & y \end{pmatrix} \tilde{v}_1, \tilde{v}_1 \rangle = \langle \tau_1(x - zy^{-1}z^*)\, (\det y)^{\kappa_p} \cdot \tilde{v}_1, \tilde{v}_1 \rangle.$$

The other identity

$$\begin{pmatrix} x & z \\ z^* & y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ z^* x^{-1} & 1 \end{pmatrix} \begin{pmatrix} x & 0 \\ 0 & y - z^* x^{-1} z \end{pmatrix} \begin{pmatrix} 1 & x^{-1} z \\ 0 & 1 \end{pmatrix}$$

has the roles of upper-triangular and lower-triangular reversed, and also τ_2 appears as τ_2^{-1}. Let $\tilde{\tau}_2$ be the irreducible representation of $GL(p+q, \mathbb{C})$ with extreme weight vector $\tilde{v}_2$ with weight

$$\tilde{\tau}_2 \begin{pmatrix} t_1 & & 0 \\ & \ddots & \\ * & & t_{p+q} \end{pmatrix} \tilde{v}_2 = t_1^{\lambda_1} \cdots t_{p+q}^{\lambda_{p+q}} \cdot \tilde{v}_2$$

where we take

$$\lambda_1 = \lambda_2 = \cdots = \lambda_p = \lambda_{p+1}.$$

Then for *lower*-triangular D in $GL(q, \mathbb{C})$

$$\tilde{\tau}_2 \begin{pmatrix} 1_p & 0 \\ 0 & D \end{pmatrix} \cdot \tilde{v}_2 = t_{p+1}^{\lambda_{p+1}} \cdots t_{p+q}^{\lambda_{p+q}} \cdot \tilde{v}_2$$

Thus, a copy of τ_2 containing $\tilde{v}_2$ lies inside the restriction of $\tilde{\tau}_2$ to $GL(q, \mathbb{C})$. Then

$$\langle \tilde{\tau}_2^{-1} \begin{pmatrix} x & z \\ z^* & y \end{pmatrix} \tilde{v}_2, \tilde{v}_2 \rangle = \langle \tilde{\tau}_2^{-1} \left[\begin{pmatrix} 1 & 0 \\ z^* x^{-1} & 1 \end{pmatrix} \begin{pmatrix} x & 0 \\ 0 & y - z^* x^{-1} z \end{pmatrix} \begin{pmatrix} 1 & x^{-1} z \\ 0 & 1 \end{pmatrix} \right] \tilde{v}_2, \tilde{v}_2 \rangle$$

$$= \langle \tilde{\tau}_2^{-1} \begin{pmatrix} x & 0 \\ 0 & y - z^* x^{-1} z \end{pmatrix} \tilde{\tau}_2^{-1} \begin{pmatrix} 1 & 0 \\ z^* x^{-1} & 1 \end{pmatrix} \tilde{v}_2, \ \tilde{\tau}_2^{*,-1} \begin{pmatrix} 1 & x^{-1} z \\ 0 & 1 \end{pmatrix} \tilde{v}_2 \rangle$$

$$= \langle \tilde{\tau}_2^{-1} \begin{pmatrix} x & 0 \\ 0 & y - z^* x^{-1} z \end{pmatrix} \tilde{v}_2, \tilde{\tau}_2^{-1} \begin{pmatrix} 1 & 0 \\ z^* x^{-1} & 1 \end{pmatrix} \tilde{v}_2 \rangle$$

$$= \langle \tilde{\tau}_2^{-1} \begin{pmatrix} x & 0 \\ 0 & y - z^* x^{-1} z \end{pmatrix} \tilde{v}_2, \ \tilde{v}_2 \rangle = \langle \tau_2^{-1} (y - z^* x^{-1} z) \, (\det x)^{-\lambda_p} \, \tilde{v}_2, \tilde{v}_2 \rangle.$$

From these computations at last we see that it is wise to take

$$s_1 = -\lambda_p \qquad s_2 = \kappa_p.$$

We do so. Combining these two computations,

$$\langle (\tilde{\tau}_1 \otimes \tilde{\tau}_2^{-1}) \begin{pmatrix} x & z \\ z^* & y \end{pmatrix} (\tilde{v}_1 \otimes \tilde{v}_2), \ \tilde{v}_1 \otimes \tilde{v}_2 \rangle$$

$$= (\det x)^{-\lambda_p} \cdot (\det y)^{\kappa_p} \cdot \langle [\tau_1 (x - z y^{-1} z^*) \otimes \tau_2^{-1} (y - z^* x^{-1} z)] \, (\tilde{v}_1 \otimes \tilde{v}_2), \ \tilde{v}_1 \otimes \tilde{v}_2 \rangle.$$

Multiplying both sides of the latter equality by $e^{-\mathrm{tr} Z} \, (\det Z)^s$ and integrating over $Z \in C_{p+q}$ with respect to the measure $dZ / (\det Z)^{p+q}$ gives

$$\langle \Gamma_{p+q}(\tilde{\tau}_1 \otimes \tilde{\tau}_2^{*,-1}, s)\, (\tilde{v}_1 \otimes \tilde{v}_2),\ \tilde{v}_1 \otimes \tilde{v}_2 \rangle =$$

$$\langle \int_{C_{p+q}} e^{-\mathrm{tr} Z} \frac{(\det y)^{\kappa_p}}{(\det x)^{\lambda_p}} \tau_1(x - zy^{-1}z^*) \otimes \tau_2^{-1}(y - z^* x^{-1} z)\, \frac{(\det Z)^s dZ}{(\det Z)^{p+q}} (\tilde{v}_1 \otimes \tilde{v}_2), \tilde{v}_1 \otimes \tilde{v}_2 \rangle$$

$$= \langle \Gamma_p(\tau_1, s - \lambda_p) \otimes \Gamma_q(\tau_2, s + \kappa_p) \cdot S \cdot (\tilde{v}_1 \otimes \tilde{v}_2),\ \tilde{v}_1 \otimes \tilde{v}_2 \rangle \rangle.$$

From above, we know that S is scalar, and also that $\Gamma_p(\tau_1, -\lambda_p + s)$ and $\Gamma_q(\tau_2^{*,-1}, \kappa_p + s)$ are scalar. *If* $\Gamma_{p+q}(\tilde{\tau}_1 \otimes \tilde{\tau}_2^{*,-1}, s)$ was scalar (*which is not at all evident!*), then we could write

$$S = S_s = \frac{\Gamma_{p+q}(\tilde{\tau}_1 \otimes \tilde{\tau}_2^{*,-1}, s)}{\Gamma_p(\tau_1, -\lambda_p + s)\ \Gamma_q(\tau_2^{*,-1}, \kappa_p + s)} .$$

However, again, it is not at all clear that $\Gamma_{p+q}(\tilde{\tau}_1 \otimes \tilde{\tau}_2^{*,-1}, s)$ is scalar, certainly not by the arguments used above, since the representation $\tilde{\tau}_1 \otimes \tilde{\tau}_2^{*,-1}$ is neither purely holomorphic nor purely antiholomorphic, unlike what was correctly invoked above. Nevertheless, we would need *less* to finish this general computation, since we are concerned with the evaluation of just a *single* inner product.

Leaving the general question aside for now, to reduce to the purely holomorphic or purely anti-holomorphic situation, it suffices to take either τ_1 or τ_2 as one-dimensional, so that anything not purely holomorphic or purely antiholomorphic can be subsumed in the power-of-determinant. Taking τ_2 to be scalar, we have

$$\lambda_1 = \lambda_2 = \cdots = \lambda_p = \lambda_{p+1} = \cdots = \lambda_{p+q}.$$

And with τ_2 scalar, the Γ_{p+q} gamma function *is scalar*, namely

$$\Gamma_{p+q}(\tilde{\tau}_1 \otimes \tilde{\tau}_2^{-1}, s) = \Gamma_{p+q}(\tilde{\tau}_1, s - \lambda_p)$$

$$= \pi^{(p+q)(p+q-1)/2} \ \textstyle\prod_{i=1}^{p+q} \Gamma(\kappa_i - (p + q - i) + (s - \lambda_p)).$$

Thus, in this case, we *can conclude*

$$S = \frac{\Gamma_{p+q}(\tilde{\tau}_1 \otimes \tilde{\tau}_2^{*,-1}, s)}{\Gamma_p(\tau_1, -\lambda_p + s)\ \Gamma_q(\tau_2^{*,-1}, \kappa_p + s)} = \frac{\Gamma_{p+q}(\tilde{\tau}_1, s - \lambda_p)}{\Gamma_p(\tau_1, -\lambda_p + s)\ \Gamma_q(\kappa_p - \lambda_p + s)} .$$

Expanding this into ordinary gammas, as in the special case done earlier, the net power of π is π^{pq}, and

$$S = \pi^{pq} \, \frac{\prod_{i=1}^{p+q} \Gamma(\kappa_i - (p + q - i) + s - \lambda_p)}{\prod_{i=1}^{p} \Gamma(\kappa_i - (p - i) + s - \lambda_p)\ \prod_{i=1}^{p+q} \Gamma(\kappa_p - (q - i) + s - \lambda_p)} .$$

Similarly, if τ_1 is scalar, then, thanks to the odd indexing scheme, we have

$$S = \pi^{pq} \frac{\prod_{i=1}^{p+q} \Gamma(\kappa_p - (p+q-i) + s - \lambda_i)}{\prod_{i=1}^{p} \Gamma(\kappa_p - (p-i) + s - \lambda_p) \prod_{i=1}^{p+q} \Gamma(\kappa_p - (q-i) + s - \lambda_i)} .$$

Thus, we can write a common expression applicable to both cases, namely

$$S = \pi^{pq} \frac{\prod_{i=1}^{p+q} \Gamma(\kappa_i - (p+q-i) + s - \lambda_i)}{\prod_{i=1}^{p} \Gamma(\kappa_i - (p-i) + s - \lambda_p) \prod_{i=1}^{p+q} \Gamma(\kappa_p - (q-i) + s - \lambda_i)} .$$

When evaluated at $s = 0$ this gives the expression asserted in Theorem 3.1.

□

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A Simple Proof of Rationality of Siegel–Weil Eisenstein Series

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Summary. We study rationality properties of Eisenstein series on quasi-split unitary similitude groups.

Introduction

This short article is concerned with the arithmetic properties of the most degenerate holomorphic Eisenstein series on quasi-split $2n$-dimensional unitary similitude groups $GH = GU(n,n)$ attached to totally imaginary quadratic extensions of totally real fields. This topic has been treated in detail in the literature, especially by Shimura. The general principle is that holomorphic Eisenstein series on Shimura varieties are rational over the fields of definition of their constant terms. In the range of absolute convergence this was proved in [4] by applying a version of the Manin–Drinfeld principle: by a combinatorial argument involving Satake parameters[1], one shows that the automorphic representations generated by absolutely convergent holomorphic Eisenstein series have multiplicity one in the space of automorphic forms. Hecke operators and the constant term map are rational over the appropriate reflex field, and this suffices to prove that Eisenstein series inherit the rationality of their constant terms, without any further computation.

The holomorphic Eisenstein series considered in this paper are lifted from the point boundary component of the Shimura variety associated to the quasi-split group GH. Rationality of modular forms on such a Shimura variety can be determined by looking at Fourier coefficients relative to the parabolic subgroup stabilizing the point boundary component (the *Siegel parabolic*). In the article [31] and in the two books [32] and [33], Shimura has obtained almost completely explicit formulas for the nondegenerate Fourier coefficients of the Eisenstein series. This should be enough to cover all relevant cases,

[1] I later learned this argument is equivalent to Langlands' geometric lemmas in his theory of Eisenstein series.

and indeed Shimura obtains applications to special values of standard L-functions of unitary groups. However, Shimura makes special hypotheses on the sections defining the Eisenstein series (equivalently, on the constant terms) that are not natural from the point of view of representation theory. In the present article we consider Eisenstein series attached to Siegel–Weil sections. These are defined representations—theoretically in terms of the local theta correspondence between $GU(n,n)$ and the unitary group $U(V)$ of a totally positive-definite Hermitian space V. Suppose $\dim V = m \geq n$. The Siegel–Weil Eisenstein series are then holomorphic values of the Eisenstein series in the (closed) right-half plane determined by the functional equation. Ichino, following the techniques of Kudla and Rallis, has recently proved the Siegel–Weil formula in this setting: the Siegel–Weil Eisenstein series is the theta lift of the constant function 1 on the adeles of $U(V)$, and a simple formula relates the constant term of the Siegel–Weil Eisenstein series to the Schwartz function defining the theta kernel in the Schrödinger model. For $\dim V < n$, the theta lift of 1 is identified with a residue of an explicit Eisenstein series, and is again a holomorphic automorphic form.

The Siegel–Weil residues, obtained when $\dim V < n$, are clearly *singular modular forms* in that their Fourier coefficients are supported on singular matrices. A special case of a theorem of J.-S. Li shows that the corresponding automorphic representations have multiplicity one. The proof of rationality now fits in a few lines. First apply Li's multiplicity one theorem and the Manin–Drinfeld principle to show that the holomorphic residual Eisenstein series on $GU(n,n)$ when $\dim V = 1$ – call these *rank one theta lifts* – inherit the rationality of their constant terms. Now let $m = \dim V \geq n$ and apply this result to $GU(nm,nm)$. This group contains $GU(n,n) \times U(V)$ (we actually work with a semi-direct product) as a subgroup. The Siegel–Weil Eisenstein series lifted from $U(V)$ to $GU(n,n)$ is obtained by integrating the rank one theta lift on $GU(nm,nm)$ over the adeles (mod principal adeles) of $U(V)$ and restricting to $GU(n,n)$. These operations are rational, and moreover are compatible with a rational map on constant terms, and this completes the proof. As a byproduct of the proof we obtain a complete description of certain spaces of classical modular forms of low weight, viewed as sections of explicit automorphic vector bundles, in terms of theta series, with a characterization of the rational elements in this space (cf. Corollaries 2.4.3 and 2.4.4).

It might be thought that even that sketch is too long. The non-degenerate Fourier coefficients of Siegel–Weil Eisenstein series can be calculated very simply in terms of the moment map. Indeed, the comparison of this calculation with Shimura's calculation for the Eisenstein series underlies Ichino's proofs of the extended Siegel–Weil formula, as it did the Kudla–Rallis extension of the Siegel–Weil formula for orthogonal-symplectic dual reductive pairs. In fact, this calculation is perfectly sufficient for determining rationality up to roots of unity. The problem is that the oscillator representation on finite adèles only becomes rational over the field generated by the additive character used to define it. In the end the choice of additive character doesn't really matter, but it enters into the calculations. In the present approach, the explicit calculations have already been carried out in the proof of the Siegel–Weil formula.

Although the ideas of the proof fit in a few lines, the paper has stretched to occupy over twenty pages. Most of this consists of notation and references to earlier work. Our use of similitude groups also introduces complications, since only unitary groups are treated in the literature. In fact, Ichino's Siegel–Weil formula does not quite extend to the full adelic similitude group, but only to a subgroup of index two, denoted $GU(n,n)(\mathbf{A})^+$. The values of Eisenstein series on the complement of $GU(n,n)(\mathbf{A})^+$ in the range of interest – to the right of the center of symmetry, but to the left of the half-plane of absolute convergence – are related to values of so-called *incoherent* Eisenstein series, or to an as yet unknown "second-term identity" in the Kudla–Rallis version of the Siegel–Weil formalism. The upshot is that we only obtain rationality of Siegel–Weil Eisenstein series over a specific quadratic extension of $\mathbb{Q}$. Adapting the arguments to $GU(n,n)(\mathbf{A})^+$ involves nothing complicated but adds considerably to the length of the paper.

Basic properties of the degenerate principal series and associated holomorphic Eisenstein series are recalled in Section 1. The first half of Section 2 recalls the theory of automorphic forms of low rank and the analogous local theory, due to Howe and Li, and uses Li's results to show that the Eisenstein series of interest to us occur with multiplicity one. These results are applied to the realization of Eisenstein series as sections of automorphic vector bundles on Shimura varieties. The rationality theorem is then derived in Section 3 from the multiplicity one property and Ichino's extension of the Siegel–Weil formula.

The paper concludes in Section 4 with applications of the rationality theorem to special values of L-functions, – up to scalars in the quadratic extension mentioned above – following [6]. This was, of course, the main motivation for the present article. For scalar weights, these results are probably all contained in [32]. Our conclusions are undoubtedly less general in some respects than Shimura's, since the use of Siegel–Weil Eisenstein series removes half the degrees of freedom in the choice of a twisting character. However, the special values we can treat are sufficient for applications to period relations anticipated in [7]. These matters will be discussed in a subsequent paper.

The reader is advised that the point boundary stratum of the Shimura variety in the applications to Siegel Eisenstein series in [4, §8] and [6] is slightly inconsistent with the general formalism for boundary strata described in [4, §6]. In fact, the latter was treated within the framework of Shimura data, as defined in Deligne's Corvallis article [1]. However, as Pink realized in [27], this is inadequate for the boundary strata. The problem is that the connected components of the zero-dimensional Shimura variety attached to $\mathbb{G}_m$, with the norm map, are all defined over the maximal totally real abelian extension of $\mathbb{Q}$, whereas the boundary points of the Shimura variety attached to $GU(n,n)$ (or $GL(2)$, for that matter) are in general defined over the full cyclotomic field, as they correspond to level structures on totally degenerate abelian varieties (i.e., on powers of $\mathbb{G}_m$. In fact, the proof of arithmeticity of Eisenstein series in [4], quoted in [6], made implicit use of Pink's formalism, although the author did not realize it at the time. For example, the reciprocity law [6, (3.3.5.4)] is correct, but is consistent with Pink's formalism rather than the framework of [4, §6]. This reciprocity law is used (implicitly) in the proof of [6, Lemma 3.5.6], which is the only place the precise determination of arithmetic Eisenstein series is applied. The discussion following Corollary 2.4.4 appeals explicitly to Pink's formalism.

I thank Atsushi Ichino for several helpful exchanges, and for providing me with the manuscript of [16]. I also thank Paul Garrett for providing an updated and expanded version of his calculation of the archimedean zeta integrals, and for allowing me to include his calculation as an appendix. All of the substantial technical problems involved in the proofs of the main theorems of the present paper were in fact solved by Ichino and Garrett. Finally, I thank the referee, whose exceptionally close reading has vastly improved the text.

0 Preliminary notation

Let E be a totally real field of degree d and $\mathcal{K}$ a totally imaginary quadratic extension of E. Let V be an n-dimensional $\mathcal{K}$-vector space, endowed with a non-degenerate Hermitian form $\langle \cdot, \cdot \rangle_V$, relative to the extension $\mathcal{K}/E$. We let Σ_E, resp. $\Sigma_{\mathcal{K}}$, denote the set of complex embeddings of E, resp. $\mathcal{K}$, and choose a CM-type $\Sigma \subset \Sigma_{\mathcal{K}}$, i.e., a subset which upon restriction to E is identified with Σ_E. Complex conjugation in $Gal(\mathcal{K}/E)$ is denoted c.

The Hermitian pairing $\langle \ , \ \rangle_V$ defines an involution on the algebra $End(V)$ via

$$\langle a(v), v' \rangle_V = \langle v, a(v') \rangle_V, \tag{0.1}$$

and this involution extends to $End(V \otimes_{\mathbb{Q}} R)$ for any $\mathbb{Q}$-algebra R. We define $\mathbb{Q}$-algebraic groups $U(V) = U(V, \langle \ , \ \rangle_V)$ and $GU(V) = GU(V, \langle \ , \ \rangle_V)$ over $\mathbb{Q}$ such that, for any $\mathbb{Q}$-algebra R,

$$U(V)(R) = \{g \in GL(V \otimes_{\mathbb{Q}} R) \mid g \cdot (g) = 1\}; \tag{0.2}$$

$$GU(V)(R) = \{g \in GL(V \otimes_{\mathbb{Q}} R) | g \cdot (g) = \nu(g) \text{ for some } \nu(g) \in (E \otimes R)^\times\}. \tag{0.3}$$

Thus $GU(V)$ admits a homomorphism $\nu : GU(V) \to R_{E/\mathbb{Q}}\mathbb{G}_m$ with kernel $U(V)$. There is an algebraic group $U_E(V)$ over E such that $U(V) \xrightarrow{\sim} R_{E/\mathbb{Q}}U_E(V)$, where $R_{E/\mathbb{Q}}$ denotes Weil's restriction of scalars functor. This isomorphism identifies automorphic representations of $U(V)$ and $U_E(V)$.

All constructions relative to Hermitian vector spaces carry over without change to skew-Hermitian spaces.

The quadratic Hecke character of $\mathbf{A}_E^\times$ corresponding to the extension $\mathcal{K}/E$ is denoted

$$\varepsilon_{\mathcal{K}/E} : \mathbf{A}_E^\times / E^\times N_{\mathcal{K}/E} \mathbf{A}_{\mathcal{K}}^\times \xrightarrow{\sim} \pm 1.$$

For any Hermitian or skew-Hermitian space, let

$$GU(V)(\mathbf{A})^+ = \ker \varepsilon_{\mathcal{K}/E} \circ \nu \subset GU(V)(\mathbf{A}). \tag{0.4}$$

For any place v of E, we let $GU(V)_v^+ = GU(V)(E_v) \cap GU(V)(\mathbf{A})^+$. If v splits in $\mathcal{K}/E$, then $GU(V)_v^+ = GU(V)(E_v)$; otherwise $[GU(V)(E_v) : GU(V)_v^+] = 2$, and $GU(V)_v^+$ is the kernel of the composition of ν with the local norm residue map. We define $GU(V)^+(\mathbf{A}) = \prod'_v GU(V)_v^+$ (restricted direct product), noting the position of the superscript; we have

$$GU(V)(E) \cdot GU(V)^+(\mathbf{A}) = GU(V)(\mathbf{A})^+. \tag{0.5}$$

1 Eisenstein series on unitary similitude groups

1.1 Notation for Eisenstein series

The present section is largely taken from [6, §3] and [7, §I.1]. Let E and $\mathcal{K}$ be as in Section 0. Let $(W, \langle\ ,\ \rangle_W)$ be any Hermitian space over $\mathcal{K}$ of dimension n. Define $-W$ to be the space W with Hermitian form $-\langle\ ,\ \rangle_W$, and let $2W = W \oplus (-W)$. Set

$$W^d = \{(v, v) \,|\, v \in W\}, \qquad W_d = \{(v, -v) \,|\, v \in W\}.$$

These are totally isotropic subspaces of $2W$. Let P (resp. GP) be the stabilizer of W^d in $U(2W)$ (resp. $GU(2W)$). As a Levi component of P we take the subgroup $M \subset U(2W)$ which is stablizer of both W^d and W_d. Then $M \simeq GL(W^d) \xrightarrow{\sim} GL(W)$, and we let $p \mapsto A(p)$ denote the corresponding homomorphism $P \to GL(W)$. Similarly, we let $GM \subset GP$ be the stabilizer of both W^d and W_d. Then $A \times \nu : GM \to GL(W) \times R_{E/\mathbb{Q}}\mathbb{G}_m$, with A defined as above, is an isomorphism. There is an obvious embedding $U(W) \times U(W) = U(W) \times U(-W) \hookrightarrow U(2W)$.

In this section we let $H = U(2W)$, viewed alternatively as an algebraic group over E or, by restriction of scalars, as an algebraic group over $\mathbb{Q}$. The individual groups $U(W)$ will reappear in Section 4. We choose a maximal compact subgroup $K_\infty = \prod_{v \in \Sigma_E} K_v \subset H(\mathbb{R})$; specific choices will be determined later. We also let $GH = GU(2W)$.

Let v be any place of E, $|\cdot|_v$ the corresponding absolute value on $\mathbb{Q}_v$, and

$$\delta_v(p) = |N_{\mathcal{K}/E} \circ \det(A(p))|_v^{\frac{n}{2}} |\nu(p)|^{-\frac{1}{2}n^2},\ p \in GP(E_v). \tag{1.1.1}$$

This is the local modulus character of $GP(E_v)$. The adelic modulus character of $GP(\mathbf{A})$, defined analogously, is denoted $\delta_{\mathbf{A}}$. Let χ be a Hecke character of $\mathcal{K}$. We view χ as a character of $M(\mathbb{A}_E) \xrightarrow{\sim} GL(W^d)(\mathbb{A}_E)$ via composition with det. For any complex number s, define

$$\begin{aligned}\delta^0_{P,\mathbf{A}}(p,\chi,s) &= \chi(\det(A(p)))\cdot|N_{\mathcal{K}/E}\circ\det(A(p))|_v^s|\nu(p)|^{-ns}\\ \delta_{\mathbf{A}}(p,\chi,s) &= \delta_{\mathbf{A}}(p)\delta^0_{P,\mathbf{A}}(p,\chi,s)\\ &= \chi(\det(A(p)))\cdot|N_{\mathcal{K}/E}\circ\det(A(p))|_v^{\frac{n}{2}+s}|\nu(p)|^{-\frac{1}{2}n^2-ns}.\end{aligned}$$

The local characters $\delta_{P,v}(\cdot,\chi,s)$ and $\delta^0_{P,v}(\cdot,\chi,s)$ are defined analogously.

Let σ be a real place of E. Then $H(E_\sigma)\xrightarrow{\sim}U(n,n)$, the unitary group of signature (n,n). As in [6, 3.1], we identify $U(n,n)$, resp. $GU(n,n)$, with the unitary group (resp. the unitary similitude group) of the standard skew-Hermitian matrix $\begin{pmatrix}0 & I_n\\ -I_n & 0\end{pmatrix}$. This identification depends on the choice of extension $\tilde{\sigma}$ of σ to an element of the CM-type Σ. We also write $GU(n,n)_\sigma$ to draw attention to the choice of σ. Let $K(n,n)_\sigma = U(n)\times U(n)\subset U(n,n)_\sigma$ in the standard embedding, $GK(n,n)_\sigma = Z\cdot K_{n,n}$ where Z is the center of $GU(n,n)$, and let $X_{n,n} = X_{n,n,\sigma} = GU(n,n)_\sigma/GK(n,n)_\sigma$, $X^0_{n,n} = U(n,n)/K(n,n)_\sigma$ be the corresponding symmetric spaces. The space $X^0_{n,n}$, which can be realized as a tube domain in the space $M(n,\mathbb{C})$ of complex $n\times n$ matrices, is naturally a connected component of $X_{n,n}$; more precisely, the identity component $GU(n,n)^+$ of elements with positive similitude factor stabilizes $X^0_{n,n}$ and identifies it with $GU(n,n)^+_\sigma/GK(n,n)_\sigma$. Writing $g\in GU(n,n)$ in block matrix form

$$g=\begin{pmatrix}A & B\\ C & D\end{pmatrix}$$

with respect to bases of W^d_σ and $W_{d,\sigma}$, we identify GP with the set of $g\in GU(n,n)$ for which the block $C=0$. In the tube domain realization, the canonical automorphy factor associated to GP and $GK(n,n)_\sigma$ is given as follows: if $\tau\in X_{n,n}$ and $g\in GU(n,n)^+$, then the triple

$$J(g,\tau)=C\tau+D,\ J'(g,\tau)=\bar{C}^t\tau+\bar{D},\nu(g) \tag{1.1.2}$$

defines a canonical automorphy factor with values in $GL(n,\mathbb{C})\times GL(n,\mathbb{C})\times GL(1,\mathbb{R})$ (note the misprint in [6, 3.3]).

Let $\tau_0\in X^+_{n,n}$ denote the unique fixed point of the subgroup $GK(n,n)_\sigma$. Given a pair of integers (μ,κ), we define a complex valued function on $GU(n,n)^+$:

$$J_{\mu,\kappa}(g)=\det J(g,\tau_0)^{-\mu}\cdot\det(J'(g,\tau_0))^{-\mu-\kappa}\cdot\nu(g)^{n(\mu+\kappa)}. \tag{1.1.3}$$

More generally, let $GH(\mathbb{R})^+$ denote the identity component of $GH(\mathbb{R})$, and let

$$GK(n,n)=\prod_\sigma GK(n,n)_\sigma,\quad K(n,n)=\prod_\sigma K(n,n)_\sigma.$$

Define $\mathbf{J}_{\mu,\kappa}: GH(\mathbb{R})^+\to\mathbb{C}^\times$ by

$$\mathbf{J}_{\mu,\kappa}((g_\sigma)_{\sigma\in\Sigma_E})=\prod_{\sigma\in\Sigma_E}J_{\mu,\kappa}(g_\sigma). \tag{1.1.4}$$

Remark. We can also let μ and κ denote integer-valued functions on Σ and define analogous automorphy factors. The subsequent theory remains valid provided the value $2\mu(\sigma)+\kappa(\sigma)$ is independent of σ. However, we will only treat the simpler case here.

In what follows, we sometimes write $GU(n,n)$ instead of GH to designate the quasi-split unitary similitude group of degree $2n$ over E or any of its completions. Let $N \subset P \subset GP$ be the unipotent radical, so that $P = M\cdot N$, $GP = GM \cdot N$.

1.2 Formulas for the Eisenstein series

Consider the induced representation

$$I_n(s,\chi) = \mathrm{Ind}(\delta^0_{P,\mathbf{A}}(p,\chi,s)) \xrightarrow{\sim} \otimes_v I_{n,v}(\delta^0_{P,v}(p,\chi,s)), \qquad (1.2.1)$$

the induction being normalized; the local factors I_v, as v runs over places of E, are likewise defined by normalized induction. Explicitly, $I_n(s,\chi)$ is the set

$$\{f : H(\mathbf{A}) \to \mathbb{C} \mid f(pg) = \delta_{P,\mathbf{A}}(p,\chi,s)f(g), p \in P(\mathbf{A}), g \in H(\mathbf{A})\}. \quad (1.2.2)$$

With this normalization the maximal E-split torus in the center of GH acts by a unitary character. At archimedean places we assume our sections to be K_∞-finite. For a section $\phi(h,s;\chi) \in I_n(s,\chi)$ (cf. [7, I.1]) we form the Eisenstein series

$$E(h,s;\phi,\chi) = \sum_{\gamma \in P(E)\backslash U(2W)(E)} \phi(\gamma h, s;\chi) \qquad (1.2.3)$$

If χ is unitary, this series is absolutely convergent for $\mathrm{Re}(s) > \frac{n}{2}$, and it can be continued to a meromorphic function on the entire plane. Let $m \geq n$ be a positive integer, and assume

$$\chi|_{\mathbf{A}_E^\times} = \varepsilon_{\mathcal{K}}^m. \qquad (1.2.4)$$

Then the main result of [34] states that the possible poles of $E(h,s;\phi,\chi)$ are all simple. Moreover, those poles in the right-half plane $Re(s) \geq 0$ can only occur at the points in the set

$$\frac{n-\delta-2r}{2}, \qquad r = 0,\ldots,[\frac{n-\delta-1}{2}], \qquad (1.2.5)$$

where $\delta = 0$ if m is even and $\delta = 1$ if m is odd. [2] We will be concerned with the values of $E(h,s_0;\phi,\chi)$ for s_0 in the set indicated in (1.2.5) when the Eisenstein series is holomorphic at s_0.

[2]Tan's main theorem lists the possible poles of the Eisenstein series multiplied by the normalizing factor he denotes b_n^*, a product of abelian L-functions. The normalizing factor has no poles to the right of the unitary axis, $Re(s) = 0$.

We write $I_n(s,\chi) = I_n(s,\chi)_\infty \otimes I_n(s,\chi)_f$, the factorization over the infinite and finite primes, respectively. Define

$$\alpha = \chi \cdot |N_{\mathcal{K}/E}|^{\frac{\kappa}{2}}.$$

We follow [6, section 3.3] and suppose the character χ has the property that

$$\alpha_\sigma(z) = z^\kappa,\ \alpha_{c\sigma}(z) = 1, z \in \mathcal{K}_\sigma^\times\ \forall \sigma \in \Sigma. \tag{1.2.6}$$

Then the function $\mathbf{J}_{\mu,\kappa}$, defined above, belongs to

$$I_n(\mu - \frac{n}{2}, \alpha)_\infty = I_n(\mu + \frac{\kappa - n}{2}, \chi)_\infty \otimes |\nu|_\infty^{\frac{n\kappa}{2}}$$

(cf. [6, (3.3.1)]). More generally, let

$$\mathbf{J}_{\mu,\kappa}(g, s + \mu - \frac{n}{2}) = \mathbf{J}_{\mu,\kappa}(g)|\det(J(g,\tau_0) \cdot J'(g,\tau_0))|^{-s} \in I_n(s,\alpha)_\infty.$$

When $E = \mathbb{Q}$, these formulas just reduce to the formulas in [6].

1.3 Holomorphic Eisenstein series

The homogeneous space $X_{n,n}^d$ can be identified with a $GH(\mathbb{R})$-conjugacy class of homomorphisms of algebraic groups over $\mathbb{R}$:

$$h : R_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_m)_{\mathbb{C}} \to GH_{\mathbb{R}}. \tag{1.3.1}$$

There is a unique conjugacy class with the property that the composition of the map (1.3.1) with any $h \in X_{n,n}^d$ induces an E-linear Hodge structure of type $(0,-1)+(-1,0)$ on $R_{\mathcal{K}/\mathbb{Q}}2W \otimes_{\mathbb{Q}} \mathbb{R}$. The chosen subgroup $GK(n,n) \subset GH(\mathbb{R})$ is the stabilizer (centralizer) of a unique point $h_0 \in X_{n,n}^d$. Corresponding to h_0 is a Harish-Chandra decomposition

$$Lie(GH)_{\mathbb{C}} = Lie(GK(n,n)) \oplus \mathfrak{p}^+ \oplus \mathfrak{p}^-$$

where $\mathfrak{p}^+$ and $\mathfrak{p}^-$ are isomorphic respectively to the holomorphic and anti-holomorphic tangent spaces of $X_{n,n}^d$ at h_0.

Let π_∞ be an admissible $(Lie(H), K(n,n))$-module. By a *holomorphic vector* in π_∞ we mean a vector annihilated by $\mathfrak{p}^-$. A *holomorphic representation* (called antiholomorphic in [6]) of $H(\mathbb{R})$ is a $(Lie(H), K(n,n))$-module generated by a holomorphic vector. The same terminology is used for admissible $(Lie(GH), GK(n,n))$-modules. As in [6, (3.3)], the element

$$\mathbf{J}_{\mu,\kappa} \in I_n(\mu + \frac{\kappa - n}{2}, \chi)_\infty \otimes |\nu|_\infty^{\frac{n\kappa}{2}}$$

is a holomorphic vector and generates an irreducible $(Lie(GH), GK(n,n))$-submodule $\mathbb{D}(\mu,\kappa)$, unitarizable up to a twist and necessarily holomorphic, which is a free $U(\mathfrak{p}^+)$-module provided $\mu + \frac{\kappa - n}{2} \geq 0$.

In our normalization, as in [19] and [16], the line $Re(s) = 0$ is the line of symmetry of the functional equation of the Eisenstein series. We will be working with an auxiliary definite Hermitian space V of dimension m. It will always be assumed that

$$m \equiv \kappa \pmod 2, \quad \mu = \frac{m-\kappa}{2}. \tag{1.3.2}$$

When $m < n$, so that $\mu + \frac{\kappa-n}{2} < 0$ is to the left of the center of symmetry of the functional equation,

$$\mathbb{D}(\mu,\kappa) \subset I_n(\mu + \frac{\kappa-n}{2}, \chi)_\infty \otimes |\nu|_\infty^{\frac{n\kappa}{2}}$$

is still unitarizable (up to a twist) but is a torsion $U(\mathfrak{p}^+)$-module (a singular holomorphic representation, cf. [17]). The lowest $GK(n,n)$-type of $\mathbb{D}(\mu,\kappa)$ is in any case given at each archimedean place v of E by the highest weight

$$\Lambda(-\mu,\kappa) = (\mu+\kappa, \mu+\kappa, \ldots, \mu+\kappa; -\mu, \ldots, -\mu; n\kappa) \tag{1.3.3}$$

(cf. [6, (3.3.2)]). Note that m can be recovered from $\Lambda(-\mu,\kappa)$. When $m = 0$, $\mathbb{D}(\mu,\kappa)$ is the one-dimensional module associated to the character $\det^{\frac{\kappa}{2}} \otimes \det^{\frac{\kappa}{2}}$. We will be most concerned with the case $m = 1$.

Let $\tilde{\mathbb{D}}(\mu,\kappa)$ be the universal holomorphic module with lowest $GK(n,n)$-type $\Lambda(-\mu,-\kappa)$:

$$\tilde{\mathbb{D}}(\mu,\kappa) = U(Lie(GH)) \otimes_{U(Lie(GK(n,n))\oplus\mathfrak{p}^-)} \mathbb{C}_{-\mu,-\kappa} \tag{1.3.4}$$

where $GK(n,n)$ acts by the character $\Lambda(-\mu,-\kappa)$ on $\mathbb{C}_{-\mu,-\kappa}$. Then $\tilde{\mathbb{D}}(\mu,\kappa)$ is a generalized Verma module, hence has a unique non-trivial irreducible quotient, which is necessarily $\mathbb{D}(\mu,\kappa)$. The same notation is used for the restrictions of these holomorphic modules to $(Lie(H), K(n,n))$.

1.3.5

The analytic continuation of the Eisenstein series defines a meromorphic map $I_n(s,\alpha) \to \mathcal{A}(H)$, where $\mathcal{A}(H)$ is the space of automorphic forms on $H(\mathbf{A})$. Assuming it is intertwining on the subspace

$$\mathbb{D}(\mu,\kappa) \otimes I_n(\mu + \frac{\kappa-n}{2}, \alpha)_f,$$

its image is tautologically generated by holomorphic vectors, namely the image of the holomorphic vector in $\mathbb{D}(\mu,\kappa)$ tensored with an arbitrary section at the finite places.

The Siegel–Weil formula, together with the characterization of Siegel–Weil sections for positive-definite unitary groups in Lemma 2.2.4, shows that the Siegel–Weil Eisenstein series for positive-definite unitary groups are always holomorphic.

2 Rank one representations of $GU(n,n)$

2.1 Local results for unitary groups

For the time being, G is either GH or H. Unless otherwise indicated, we always assume $n > 1$. Let v be a place of E that does not split in $\mathcal{K}$, E_v the completion. Choose a non-trivial additive character $\psi : E_v \to \mathbb{C}^\times$. The subgroup $N(E_v) \subset G(E_v)$ is naturally isomorphic with its own Lie algebra, which we identify with the vector group of $n \times n$ Hermitian matrices over $\mathcal{K}_v$, and we can then identify $N(E_v)$ with its own Pontryagin dual by the pairing

$$N(E_v) \times N(E_v) \to \mathbb{C}^\times; (n, n') = \psi_n(n') := \psi(Tr(n \cdot n')). \qquad (2.1.1)$$

Formula (2.1.1) defines the character ψ_n, which we also use to denote the space $\mathbb{C}$ on which $N(E_v)$ acts through the character ψ_n. For the remainder of this section we write N, P, M, G, and so on for $N(E_v), P(E_v)$, etc.

Let π be an irreducible admissible representation of G. The N-spectrum of π is the set of $n \in N$ such that $Hom_N(\pi, \psi_n) \neq 0$. The representation π is said to be *of rank one* if its N-spectrum is contained in the subset of matrices of rank one.

The subgroup $GM \subset GP(E_v)$ acts on N by conjugation. For any $\nu \in E_v^\times$, let

$$d(\nu) = \begin{pmatrix} 1 & 0 \\ 0 & \nu \end{pmatrix} \in GH.$$

The following facts are well-known and due to Howe and Li; see [24, §3], for statements and references.

Lemma 2.1.2. *(a) Let $G = GH$ (resp. H). For any π, the N-spectrum of π is a union of GM- (resp. M-) orbits. If π is of rank one, its N-spectrum contains a single GM- (resp. M-) orbit of rank one.*
(b) Let $\alpha \in E_v^\times$ be an element that is not a norm from $\mathcal{K}_v$. The set of rank one matrices in N is the union of two M-orbits $\mathcal{O}_1$ and $\mathcal{O}_\alpha$, represented respectively by the matrices with 1 and α in the upper left-hand corner and zeros elsewhere.
(c) The set of rank one matrices in N is a single GM-orbit; the element d_α of GH exchanges $\mathcal{O}_1$ and $\mathcal{O}_\alpha$.

Thus the irreducible representations π of H of rank one can be classified as type 1 or type α, as their spectrum contains $\mathcal{O}_1$ or $\mathcal{O}_\alpha$. This classification is relative to the choice of ψ.

Let ε denote the quadratic character of $E_v^\times$ associated to $\mathcal{K}_v$. Let V be any Hermitian space over $\mathcal{K}_v$ of dimension m. Let $\chi_v : \mathcal{K}_v^\times \to \mathbb{C}^\times$ be a character restricting to ε^m on $E_v^\times$. The map $x \mapsto x/c(x)$ defines an isomorphism

$$\mathcal{K}_v^\times / E_v^\times \xrightarrow{\sim} U(1)_v = \ker N_{\mathcal{K}_v/E_v} \subset \mathcal{K}^\times;$$

$U(1)_v$ is also the unitary group of any one-dimensional Hermitian space. In particular, any character β of $\mathcal{K}_v^\times/E_v^\times$ defines a character $\tilde{\beta}$ of $U(1)_v$:

$$\beta(x) = \tilde{\beta}(x/c(x)),$$

and the map $\beta \mapsto \tilde{\beta}$ is a bijection. Multiplication makes the set of χ_v with given restriction to $E_v^\times$ into a principal homogeneous space under the group of characters of $U(1)_v$. Choose a second character $\chi'_v : \mathcal{K}_v^\times/E_v^\times \to \mathbb{C}^\times$ be a second character, which may be trivial. As in [11], the pair (χ_v, χ'_v), together with the additive character ψ, define a Weil representation

$$\omega_{\chi_v,\chi'_v,\psi} : H \times U(V) \to Aut(\mathcal{S}(V^n)),$$

where $\mathcal{S}$ denotes the Schwartz–Bruhat space. The formulas for the action of H are given in [15]. The group $U(V)$ acts linearly on the argument when χ'_v is trivial, and in general the linear action is twisted by $\chi'_v \circ \det$, cf. (2.2.1) below. Let $R_n(V, \chi_v, \chi'_v) = R_n(V, \chi_v, \chi'_v, \psi)$ denote the maximal quotient of $\mathcal{S}(V^n)$ on which $U(V)$ acts trivially. We abbreviate $R_n(V, \chi_v) = R_n(V, \chi_v, triv_v)$.

More generally, for any irreducible representation β of $U(V)$, let

$$\Theta_{\chi_v,\chi'_v,\psi}(\beta) = \Theta_{\chi_v,\chi'_v,\psi}(U(V) \to U(n,n); \beta) \tag{2.1.3}$$

denote the maximal quotient of $\mathcal{S}(V^n) \otimes \beta$ on which $U(V)$ acts trivially; this is a representation of H.

For the remainder of this section $m = \dim V = 1$. Let V^+ denote the one-dimensional $\mathcal{K}_v$-space with Hermitian form $(z, z') = z \cdot \bar{z}'$, where $^-$ denotes Galois conjugation. Let V^- denote the same space with the Hermitian form multiplied by α. Up to isomorphism, V^+ and V^- are the only two Hermitian spaces over $\mathcal{K}_v$ of dimension 1. We let ψ_α be the character $\psi_\alpha(x) = \psi(\alpha \cdot x)$.

Proposition 2.1.4. *[20] The spaces $R_n(V^\pm, \chi_v, \psi)$ are irreducible. More precisely, $R_n(V^+, \chi_v, \psi)$ and $R_n(V^-, \chi_v, \psi)$ are the unique non-trivial irreducible quotients of the induced representation $I_n(\frac{n-1}{2}, \chi_v)$; alternatively, they are isomorphic to the unique non-trivial irreducible subrepresentations of the induced representation $I_n(\frac{1-n}{2}, \chi_v)$.*

The first part of the following theorem is equivalent to a special case of a result of Li:

Theorem 2.1.5. *(a) Every rank one representation π of H is isomorphic to a representation of the form $R_n(V^\pm, \chi_v, \chi'_v, \psi)$ for some χ_v, χ'_v, with fixed ψ. The N-spectrum of $R_n(V^+, \chi_v, \chi'_v, \psi)$ (resp. $R_n(V^-, \chi_v, \chi'_v, \psi)$) is contained in the closure of $\mathcal{O}_1$ (resp. $\mathcal{O}_\alpha$).*

(b) For any χ_v, χ'_v,

$$R_n(V^+, \chi_v, \chi'_v, \psi_\alpha) = R_n(V^-, \chi_v, \chi'_v, \psi),$$
$$R_n(V^-, \chi_v, \chi'_v, \psi_\alpha) = R_n(V^+ \chi_v, \chi'_v, \psi).$$

(c) There are natural isomorphisms

$$R_n(V^{\pm}, \chi_v, \chi'_v, \psi) \xrightarrow{\sim} \Theta_{\chi_v, triv_v, \psi}(U(V^{\pm}) \to U(n,n); (\chi'_v)^{-1});$$

$$R_n(V^{\pm}, \chi_v \cdot \beta, \chi'_v, \psi) \xrightarrow{\sim} R_n(V^{\pm}, \chi_v, \chi'_v, \psi) \otimes \tilde{\beta} \circ \det$$

if β is trivial on $E_v^{\times}$.

(d) No two representations $R_n(V^{\pm}, \chi_v, \chi'_v, \psi)$ and $R_n(V^{\pm}, \chi_v, \chi''_v, \psi)$ are isomorphic if $\chi'_v \neq \chi''_v$.

Proof. The first part of (a) follows from [23, Theorem 4.8], and the second part is a special case of [23, Lemma 4.4]. Actually, Li's theorem identifies rank one representations as theta lifts only up to character twists, but (c) shows that all such twists are obtained by varying χ'_v. Assertion (b) is standard (cf. [26, p. 36 (2)]), while (c) follows from properties of splittings proved in [18] and recalled in [11], specifically [11, (1.8)]. Finally, (d) follows from the first formula of (c) and Howe duality for the dual reductive pair $(U(1), U(n,n))$. □

In accordance with Theorem 2.1.5(a), we say V^+ (resp. V^-) *represents* $\mathcal{O}_1$ (resp. $\mathcal{O}_\alpha$). Note that this correspondence between orbits and Hermitian spaces depends only on ψ and not on the choice of characters χ_v, χ'_v.

The corresponding results at real places were studied by Lee and Zhu [22]. We state a version of their results for similitude groups in the following section.

2.1.6 Split places

The situation at places v that split in $\mathcal{K}/E$ is simpler in that there is no dichotomy between V^+ and V^-, there is only one orbit $\mathcal{O}_1$, and only one equivalence class of additive characters. Otherwise, Theorem 2.1.5 remains true: every rank one representation is a theta lift from $U(1) = GL(1)$. Proposition 2.1.4 is also valid in this situation: the theta lift can be identified with an explicit constituent of a degenerate principal series representation [20, Theorem 1.3] with the same parametrization as in the non-split case.

2.2 Local results for similitude groups

Let V and $U(V)$ be as in Section 2.1. Let $GH^+ \subset GH$ be the subgroup of elements h such that $\varepsilon \circ \nu(h) = 1$, i.e., the similitude of h is a norm from $\mathcal{K}_v$. Then $[GH : GH^+] \leq 2$. Define $GU(V)^+ \subset GU(V)$ analogously. Let $GU(V)$ act on H by the map β:

$$\beta(g)(h) = d(\nu(g))hd(\nu(g))^{-1},$$

with $d(\nu(g)) \in GH$ defined as in Section 2.1. As in [10, §3], the Weil representation extends to an action of the group

$$R = R_v = \{(g,h) \in GU(V) \times GH \mid \nu(h) = \nu(g)\}.$$

There is a surjective map $R \to GU(V) \ltimes H$:

$$(g,h) \mapsto (g, d(\nu(h))^{-1}h) = (g,h_1).$$

There is a representation of R on $\mathcal{S}(V^n)$ given by

$$\omega_{\chi_v,\chi'_v,\psi}(g,h_1)\Phi(x) = |\nu(g)|^{-\frac{\dim V}{2}}\chi'_v(\det(g))(\omega_{\chi_v,\chi'_v,\psi}(h_1)\Phi)(g^{-1}x). \quad (2.2.1)$$

The power $|\nu(g)|^{-\frac{\dim V}{2}}$ guarantees that the maximal E-split torus in the center of GH acts by a unitary character, as in $I_n(s,\chi)$. This is not what we need for arithmetic applications but is helpful for normalization.

The image $U(V) \subset GU(V)$ in R is a normal subgroup and $R/U(V)$ is isomorphic either to GH^+ or to GH. When $\dim V = 1$, $R \xrightarrow{\sim} GU(V) \ltimes H$ and $R/U(V) \xrightarrow{\sim} GH^+$. If β is a representation of $GU(V)$ that restricts irreducibly to $U(V)$, then $\Theta_{\chi_v,\chi'_v,\psi}(\beta)$, defined as in (2.1.3), extends to a representation of GH^+, which we denote $\Theta^+_{\chi_v,\chi'_v,\psi}(\beta)$. We let

$$\Theta_{\chi_v,\chi'_v,\psi}(\beta) = Ind_{GH^+}^{GH}\Theta^+_{\chi_v,\chi'_v,\psi}(\beta).$$

This need not be irreducible, but in the cases we consider it will be. See [5, §3] for constructions involving similitude groups.

We again restrict attention to $\dim V = 1$, and $\chi'_v = triv_v$. Then $GU(V) = \mathcal{K}^\times$, acting by scalar multiplication on V. For the character β of $GU(V)$ we just take the trivial character; then $\Theta^+_{\chi_v,triv_v,\psi}(triv)$ is an extension of $R_n(V,\chi_v)$ to a representation $R_n^+(V,\chi_v)$ of GH^+, rigged so that the maximal E_v-split torus in the center acts by a unitary character.

Proposition 2.2.2. *The representations $R_n^+(V^+,\chi_v)$ and $R_n^+(V^-,\chi_v)$ of GH^+ are inequivalent and are conjugate under the element $d(\alpha)$ of $GH - GH^+$. In particular,*

$$Ind_{GH^+}^{GH}R_n^+(V^+,\chi_v) = Ind_{GH^+}^{GH}R_n^+(V^-,\chi_v)$$

is an irreducible representation, denoted $R_n(V^\pm,\chi_v)$, of GH. It is the unique non-trivial irreducible quotient of the induced representation $I_n(\frac{n-1}{2},\chi_v)$. Alternatively, it is the unique non-trivial irreducible subrepresentation of the induced representation $I_n(\frac{1-n}{2},\chi_v)$.

We have

$$R_n(V^\pm,\chi_v) \ |_H = R_n(V^+,\chi_v) \oplus R_n(V^-,\chi_v).$$

Proof. Theorem 2.1.5 asserts that the two irreducible representations $R_n^+(V^+,\chi_v)$ and $R_n^+(V^-,\chi_v)$ have distinct N-spectra upon restriction to H. This implies the first assertion. To show the two representations are conjugate under $d(\alpha)$, we note first that $d(\alpha)$ exchanges their N-spectra (Lemma 2.1.2(c)). On the other hand, the induced representation $I_n(n-\frac{1}{2},\chi_v)$ of H extends (in more than one way) to a representation of GH; one such extension is defined in (1.1). The claim then follows from Proposition 2.1.4. The remaining assertions are then obvious, given that $I(s,\chi_v)$ has also been rigged to have unitary central character on the maximal E_v-split torus. □

2.2.3 Intertwining with induced representations

In this section $m = \dim V$ is arbitrary. The quotient $R_n(V,\chi)$ of $\mathcal{S}(V^n)$ can be constructed explicitly as a space of functions on GH^+. For any $h \in GH^+$, let $g_0 \in GU(V)$ be any element with $\nu(g_0) = \nu(h)$. For $\Phi \in \mathcal{S}(V^n)$, define

$$\phi_\Phi(h) = (\omega_{\chi_v,triv,\psi}(g_0,h)\Phi)(0). \tag{2.2.3.1}$$

This function does not depend on the choice of g_0 (cf. [10], *loc. cit.*) and belongs to the space of restrictions to GH^+ of functions in $I_n(s_0,\chi_v)$ with $s_0 = \frac{m-n}{2}$. The action of GH^+ on $I_n(s_0,\chi_v)$ by right translation extends to the natural action of GH, and in this way the function ϕ_Φ on GH^+ extends canonically to a function on GH. Since $d(\nu) \in GP$ for all $\nu \in E_v^\times$, the formula for this extension is simply

$$\phi_\Phi(h^+d(\nu)) = \delta_{P,v}(d(\nu),\chi,s_0) \cdot \phi_\Phi(d(\nu)^{-1}h^+d(\nu)), \tag{2.2.3.2}$$

which is consistent with (2.2.3.1) when $d(\nu) \in GH^+$.

Lemma 2.2.4. *Let* $\dim V^+ = m < n$. *Suppose* v *is a real place and* $\psi(x) = e^{ax}$ *with* $a > 0$. *Then* $R_n(V^+,\chi_v,\psi)$ *is a holomorphic representation of* GH_v^+. *If* $\chi_v(z) = z^\kappa/|z|^\kappa$, *then*

$$R_n(V^+,\chi_v,\psi) \otimes |\nu|_\infty^{\frac{n\kappa}{2}} \xrightarrow{\sim} \mathbb{D}(\mu,\kappa)$$

in the notation of (1.3), with $\mu = \frac{m-\kappa}{2}$.

Proof. As in Section 1.3, our representations of real groups are objects in the category of Harish-Chandra modules. This is an elementary calculation, and the result is a special case of the general results of [22], specifically the K-type calculations in Section 2, Proposition 2.1 and 2.2. □

2.3 Global multiplicity-one results

As above, we always assume $n \geq 2$ unless otherwise indicated. We fix an additive character $\psi : \mathbf{A}_E/E \to \mathbb{C}^\times$ and a Hecke character χ of $\mathcal{K}^\times$ satisfying (1.2.4), with $m = 1$.

Definition 2.3.1. *An automorphic representation* π *of* GH *or* H *is said to be locally of rank one at the place* v *of* E *if* π_v *is of rank one. The representation* π *is said to be globally of rank one if* π_v *is of rank one for all* v.

The main results on rank one representations are summarized in the following theorem. Most of the assertions are special cases of a theorem of J.-S. Li [24].

Theorem 2.3.2. *(a) An automorphic representation of* H *is globally of rank one if and only if it is locally of rank one at some place* v.

(b) Let π be an automorphic representation of H of rank one. Then there is a $\mathcal{K}$ Hermitian space V of dimension one, a character β of $U(V)(E)\backslash U(V)(\mathbf{A})$, and a Hecke character χ of $\mathcal{K}^\times$ satisfying (1.2.4), such that

$$\pi = \Theta_{\chi,triv,\psi}(U(V) \to U(n,n); \beta).$$

(c) Any automorphic representation of H of rank one occurs with multiplicity one in the space of automorphic forms on H.

(d) Suppose $\psi_v(x) = e^{a_v x}$ with $a_v > 0$ for all archimedean v. Then an automorphic representation of H of rank one is holomorphic (resp. anti-holomorphic) if and only if $V_v = V^+$ (resp. V^-) for all archimedean v.

(e) Every automorphic representation of H of rank one is contained in the space of square-integrable automorphic forms on H.

(f) Let $\pi = \otimes_v \pi_v$ be an irreducible admissible representation of H with π_v of rank one for all v. Let the orbit $\mathcal{O}_v$ be the N_v spectrum of π_v, and let V_v be the equivalence class of one-dimensional Hermitian spaces over $\mathcal{K}_v$ representing $\mathcal{O}_v$. Suppose there is no global one-dimensional Hermitian space V that localizes to V_v at each v. Then π has multiplicity zero in the space of automorphic forms.

Proof. Assertion (a) is due to Howe [14]. In view of Li's Theorem 2.1.5 (a), assertions (b), (c), and (e) are contained in Theorem A of [24]; note that any automorphic form on any $U(1)$ relative to $\mathcal{K}/E$ is necessarily square-integrable! Assertion (d) is a calculation of the local Howe correspondence (cf. [7, II (3.8)]). Finally, assertion (f) is a formal consequence of (b), and equivalently of the fact that such a π can have no non-constant Fourier coefficients. □

Corollary 2.3.3. *(a) An automorphic representation of GH is globally of rank one if and only if it is locally of rank one at some place v.*

(b) Let π be an automorphic representation of GH of rank one. Then there is a $\mathcal{K}$ Hermitian space V, $\dim V = 1$, a character β of $GU(V)(E)\backslash GU(V)(\mathbf{A})$, and a Hecke character χ of $\mathcal{K}^\times$ satisfying (1.2.4), such that $\pi = \Theta_{\chi,triv,\psi}(\beta)$.

(c) Any automorphic representation of GH of rank one occurs with multiplicity one in the space of automorphic forms on GH.

(d) Let π be an automorphic representation of GH of rank one. As representation of $Lie(GH)_{\mathbb{C}} \times GH(\mathbf{A}_f)$, π is generated by $\{\pm 1\}^{\Sigma_E}$ vectors $v_{(e_\sigma)}$, with each $e_\sigma \in \{\pm 1\}$. The vector $v_{(e_\sigma)}$ is holomorphic (resp. anti-holomorphic) at each place σ with $e_\sigma = +1$ (resp. $e_\sigma = -1$.

(e) Every automorphic representation of GH of rank one is contained in the space of essentially square-integrable automorphic forms on GH.

"Essentially square-integrable" in the above corollary means square-integrable modulo the adeles of the center, up to twist by a character.

The global automorphic representation $\Theta_{\chi,triv,\psi}(\beta)$ is isomorphic to the restricted tensor product over v of the local representations $\Theta_{\chi_v,triv,\psi_v}(\beta_v)$.

As a space of automorphic forms it is defined as in [7, §I.4]. The elements of this space are defined below. We are primarily interested in the case of trivial β. Then

$$\Theta_{\chi,triv,\psi}(triv) = R_n(V^{\pm},\chi) \stackrel{def}{=} \otimes_v R_n(V^{\pm},\chi_v) \\ \stackrel{\sim}{\longrightarrow} Ind_{GH^+(\mathbf{A})}^{GH(\mathbf{A})} \otimes_v R_n(V_v,\chi_v), \tag{2.3.4}$$

where V is any one-dimensional Hermitian space over $\mathcal{K}$ and V_v is its localization at v. The space $R_n(V^{\pm},\chi)$ may be viewed alternatively, as in the unique non-trivial irreducible $GH(\mathbf{A})$-quotient of the adelic induced representation $I_n(n-\frac{1}{2},\chi)$ or as the unique non-trivial irreducible $GH(\mathbf{A})$-submodule of $I_n(\frac{1}{2},\chi)$.

The dimension of V is now arbitrary. Let $R(\mathbf{A}_f) = \prod_v' R_v$, where $R_v \subset GH(E_v) \times GU(V_v)$ is the group defined in (2.2). Let $\Phi \in \mathcal{S}(V^n)$ and define

$$\theta_{\chi,triv,\psi}(\Phi)(g,h) = \sum_{x\in V^n(\mathcal{K})} \omega_{\chi,triv,\psi}(g,h)(\Phi)(x). \tag{2.3.5}$$

For $h \in GH^+(\mathbf{A})$, defined as in the notation section, and for $g_0 \in GU(V)(\mathbf{A})$ such that $\nu(g_0) = \nu(h)$, let

$$I_{\chi,triv,\psi}(\Phi)(h) = \int_{U(V)(E)\backslash U(V)(\mathbf{A})} \theta_{\chi,triv,\psi}(\Phi)(g_0 g,h)dg, \tag{2.3.6}$$

assuming the right-hand side converges absolutely, which will always be the case in the applications. The measure dg is the Tamagawa measure. This integral does not depend on the choice of g_0 and defines an automorphic form on $GH^+(\mathbf{A}) \cap GH(\mathbb{Q})\backslash GH^+(\mathbf{A})$ that extends uniquely to a function on

$$GH(\mathbb{Q})\backslash GH(\mathbb{Q}) \cdot GH^+(\mathbf{A}) = GH(\mathbb{Q})\backslash GH(\mathbf{A})^+,$$

also denoted $I_{\chi,triv,\psi}(\Phi)$.

In Section 3 we will realize this function in a certain space of Eisenstein series. As such, it can be extended to an automorphic form on all $GH(\mathbf{A})$. We now assume $\dim V = 1$. As abstract representation, the restriction of $\Theta_{\chi,triv,\psi}(triv)$ to $H(\mathbf{A})$ is calculated by the final assertion of Proposition 2.2.2:

$$\Theta_{\chi,triv,\psi}(triv) \mid_{H(\mathbf{A})} = \bigoplus_S \otimes_{v\notin S} R(V_v^+,\chi_v,\psi_v) \bigotimes \otimes_{v\in S} R(V_v^-,\chi_v,\psi_v), \tag{2.3.7}$$

where S runs over all finite sets of places of E. Let Θ_S denote the summand on the right-hand side of (2.3.7) indexed by S. We say Θ_S is *coherent* if there is a global V such that $V_v = V_v^-$ if and only if $v \in S$, incoherent otherwise. By the global reciprocity law for the cohomological invariant of Hermitian spaces, S is coherent if and only if $|S|$ is even.

Lemma 2.3.8. *Let F be any extension of $I_{\chi,triv,\psi}(\Phi)$ to an automorphic form on $GH(\mathbf{A})$. Then $F(h) = 0$ for $h \in GH(\mathbf{A}) - GH(\mathbf{A})^+$.*

Proof. An extension of $I_{\chi,triv,\psi}$ to $GH(\mathbf{A})$ corresponds to a homomorphism λ from

$$\Theta_{\chi,triv,\psi}(triv) = Ind_{GH^+(\mathbf{A})}^{GH(\mathbf{A})} \otimes_v R_n(V_v, \chi_v)$$

to the space of automorphic forms on $GH(\mathbf{A})$. Let Θ_S denote the summand on the right-hand side of (2.3.7) indexed by S. It follows from (2.3.2)(f) that $\lambda(\Theta_S) = 0$ for any incoherent S. The lemma now follows easily from Lemma 2.1.2(c). □

2.4 Automorphic forms on the Shimura variety

Let $\mathcal{A}(n,n)$ denote the space of automorphic forms on GH; i.e., of functions on $GH(E)\backslash GH(\mathbf{A}_E)$ satisfying the axioms of automorphic forms. Let $GH^0(\mathbf{A}_E) = GH_\infty^+ \times GH(\mathbf{A}_f)$, where $GH_\infty = GH(E \otimes_{\mathbb{Q}} \mathbb{R})$ and GH_∞^+ is the identity component, which can also be characterized as the subgroup with positive similitude factor at each real place. By "real approximation", $GH(E) \cdot GH^0(\mathbf{A}_E) = GH(\mathbf{A})$, so an automorphic form is determined by its restriction to $GH(E)^0\backslash GH^0(\mathbf{A}_E)$, where $GH(E)^0 = GH(E) \cap GH^0(\mathbf{A})_E$.

The pair $(GH, X_{n,n}^d)$ defines a Shimura variety $Sh(n,n)$, or $Sh(n,n)_{\mathcal{K}/E}$, with canonical model over its reflex field, which is easily checked to be $\mathbb{Q}$ (see Section 2.6 below). This Shimura variety does not satisfy a hypothesis frequently imposed: the maximal $\mathbb{R}$-split subgroup of the center of G is not split over $\mathbb{Q}$. Let $Z_E \subset Z_{GH}$ be the maximal $\mathbb{Q}$-anisotropic subtorus of the maximal $\mathbb{R}$-split torus; concretely, Z_E is the kernel of the norm from $R_{E/\mathbb{Q}}(\mathbb{G}_m)_E$ to $\mathbb{G}_m$. Automorphic vector bundles over $Sh(n,n)$ are indexed by irreducible representations of $GK(n,n)$ on which Z_E acts trivially.

We only need automorphic line bundles, indexed by the characters $\Lambda(\mu,\kappa)$ of $GK(n,n)$ whose inverses were defined in (1.3.3); note that Z_E acts trivially because the characters μ and κ are constant as functions of real places. Let $\mathcal{E}_{\mu,\kappa}$ be the corresponding line bundle, defined as in [6, (3.3)]. It is elementary (cf. [6, (3.3.4)]) that

$$Hom_{(Lie(GH),GK(n,n))}(\tilde{\mathbb{D}}(\mu,\kappa), \mathcal{A}(n,n)) \xrightarrow{\sim} H^0(Sh(n,n), \mathcal{E}_{\mu,\kappa}). \tag{2.4.1}$$

(In [*loc. cit.*] the right-hand side is replaced by the space of sections of the canonical extension of $\mathcal{E}_{\mu,\kappa}$ over some toroidal compactification $\widetilde{Sh(n,n)}$ of $Sh(n,n)$. Since $n > 1$, Koecher's principle guarantees these spaces are canonically isomorphic.)

Theorem 2.4.2. *Assume $\mu + \frac{\kappa-n}{2} < 0$. Then every homomorphism on the left-hand side of (2.4.1) factors through $\mathbb{D}(\mu,\kappa)$.*

Proof. Defining m by (1.3.2), the hypothesis implies that $m < n$. This theorem is essentially due to Resnikoff [28]; the current formulation is from [9]. □

Corollary 2.3.3 and Lemma 2.1.6 then imply:

Corollary 2.4.3. *Hypotheses are as in Theorem 2.4.2*

(a) The space $H^0(Sh(n,n),\mathcal{E}_{\mu,\kappa})$ is generated by the images with respect to (2.4.1) of forms in

$$\Theta^{hol}_{\chi,triv,\psi}(\beta)\otimes|\nu|_\infty^{\frac{n\kappa}{2}}$$

where β runs through characters of $GU(V)(\mathbf{A})/GU(V)(E)$ with $\beta_\infty = 1$, and where $\chi_v(z) = z^\kappa/|z|^\kappa$ for all real v.

(b) The representation of $GH(\mathbf{A}_f)$ on $H^0(Sh(n,n),\mathcal{E}_{\mu,\kappa})$ is completely reducible and multiplicity free.

Proof. The condition $\beta_\infty = 1$ implies, in view of Lemma 2.2.4, that the archimedean components of $\Theta^{hol}_{\chi,triv,\psi}(\beta)|\nu|_\infty^{\frac{n\kappa}{2}}$ are all isomorphic to $\mathbb{D}(\mu,\kappa)$. This is not possible for $\beta_\infty \neq 1$ by Howe duality; thus, only β with a trivial archimedean component contributes to the left-hand side of (2.4.1). Thus (a) follows from Corollary 2.3.3 (a) and (b) and Lemma 2.1.6. Assertion (b) is a consequence of (2.3.3) (e) and (c): the L^2 pairing defines a Hermitian (Petersson) pairing on $H^0(Sh(n,n),\mathcal{E}_{\mu,\kappa})$ with respect to which the action of $GH(\mathbf{A}_f)$ is self-adjoint. □

It follows from the theory of canonical models of automorphic vector bundles [4], [25] that the automorphic line bundle $\mathcal{E}_{\mu,\kappa}$ is defined over the field of definition $E(\mu,\kappa)$ of the conjugacy class under the normalizer of $GK(n,n)$ in GH of the character $\Lambda(\mu,\kappa)$, which is contained in the reflex field $E(\Sigma)$ of the CM-type Σ. If $\kappa \neq 0$, as will always be the case in applications, then $E(\mu,\kappa) = E(\Sigma)$. Improvement on this field of definition requires a more careful analysis of the interchange of holomorphic and anti-holomorphic forms on $Sh(n,n)$.

The isomorphism (2.4.1) implicitly depends on the choice of a basis for the fiber of $\mathcal{E}_{\mu,\kappa}$ at the fixed point of $GK(n,n)$ in $X^+_{n,n}$, called a *canonical trivialization* in [6]. This choice enters into explicit calculations with automorphic forms but is irrelevant for the present purposes. All that matters is that, with respect to this choice, which we fix once and for all, the $E(\mu,\kappa)$-rational structure on the right-hand side of (2.4.1) defines an $E(\mu,\kappa)$-rational structure on the left-hand side.

The next corollary is obvious from what has been presented thus far.

Corollary 2.4.4. *Let π_f be an irreducible representation of $GH(\mathbf{A}_f)$ that occurs in $H^0(Sh(n,n),\mathcal{E}_{\mu,\kappa})$, and let $M[\pi_f] \subset H^0(Sh(n,n),\mathcal{E}_{\mu,\kappa})$ denote the π_f-isotypic subspace. Then π_f is defined over a finite extension $E(\pi_f)$ of $E(\mu,\kappa)$, $M[\pi_f]$ is isomorphic as $GH(\mathbf{A}_f)$-module to π_f, and $\gamma(M[\pi_f]) = M([\gamma(\pi_f)])$ for any $\gamma \in Gal(\overline{\mathbb{Q}}/E(\mu,\kappa))$.*

As in [6, 3.3.5], we denote by $Sh(n,n)_{GP}$ the point boundary stratum of the minimal (Baily–Borel–Satake) compactification of $Sh(n,n)$. Our description of this stratum follows the formalism of Pink [27]. Let $(G_{h,P}, h_P^{\pm})$ be the Shimura datum associated to this boundary stratum. The group $G_{h,P}$ is a torus contained in GP, isomorphic to $\mathbb{G}_m \times \mathbb{G}_m$, as in [6] (where there is however a misprint):

$$G_{h,P} = \{g \in GP \ \mid A = aI_n, D = dI_n\} \subset GP$$

and $h_P^{\pm}$ is a homogeneous space for $G_{h,P}(\mathbb{R}) = \mathbb{R}^{\times} \times \mathbb{R}^{\times}$, consisting of two points on which (a,d) acts trivially if and only if $a > 0$. This condition covers (in Pink's sense) the $\mathbb{R}$-homomorphism $h_P : R_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_m)_{\mathbb{C}} \to G_{h,P}$ defined by [6, (3.3.5.2)]. More precisely, in [6] we have $E = \mathbb{Q}$; in general, h_P is given by [6, (3.3.5.2)] at each real place of E. The same is true for the formula defining the limit bundle $\mathcal{E}_{\mu,\kappa,GP}$: the line bundle $\mathcal{E}_{\mu,\kappa}$ extends to a line bundle on the minimal compactification, whose restriction to $Sh(n,n)_{GP}$ is denoted $\mathcal{E}_{\mu,\kappa,GP}$, associated to a character denoted $\lambda_{\mu,\kappa,n}$ on p. 143 of [6]. The exact formula does not matter.

Let $\mathfrak{f}_P : H^0(Sh(n,n), \mathcal{E}_{\mu,\kappa}) \to H^0(Sh(n,n)_{GP}, \mathcal{E}_{\mu,\kappa,GP})$ denote the Siegel Φ-operator (constant term map) for holomorphic forms, defined as in [4, §6]. In classical language, $\mathfrak{f}_P$ takes a holomorphic form of weight (μ,κ) to the constant term of its Fourier expansion relative to the tube domain realization of the universal cover of $Sh(n,n)$. It follows from the general theory [*loc. cit*] that $\mathfrak{f}_P$ is rational over $E(\mu,\kappa) = E(\Sigma)$ and intertwines the $H(\mathbf{A}_f)$ actions. Thus:

Lemma 2.4.5. *Let π_f be an irreducible admissible representation of $GH(\mathbf{A}_f)$ occurring in $H^0(Sh(n,n), \mathcal{E}_{\mu,\kappa})$ and let $\mathfrak{f}_P[\pi_f]$ denote the restriction of $\mathfrak{f}_P$ to $M[\pi_f] \subset H^0(Sh(n,n), \mathcal{E}_{\mu,\kappa})$. Then $\mathfrak{f}_P[\pi_f]$ is either zero or an isomorphism onto its image. For any $\gamma \in Gal(\overline{\mathbb{Q}}/E(\mu,\kappa))$, $\mathfrak{f}_P[\gamma(\pi_f)] = \gamma \circ \mathfrak{f}_P[\pi_f]$.*

Lemma 2.4.5 is expressed in terms of sections of automorphic vector bundles. With respect to the isomorphism (2.4.1), $\mathfrak{f}_P$ can be identified as the constant term which we denote r_P, in the theory of automorphic forms. More precisely, there is a commutative diagram (cf. [4, (8.1.2)] and [6, Lemma 3.3.5.3]):

$$\begin{array}{ccc} Hom_{(Lie(GH),GK(n,n))}(\tilde{\mathbb{D}}(\mu,\kappa), \mathcal{A}(n,n)) & \xrightarrow[\sim]{} & H^0(Sh(n,n), \mathcal{E}_{\mu,\kappa}) \\ \Big\downarrow r_P & & \mathfrak{f}_P \Big\downarrow \\ Hom_{GP(\mathbb{R})}(\lambda_{\mu,\kappa,n}^{-1}, I_{GP(\mathbf{A}_f)}^{GH(\mathbf{A}_f)} \mathcal{A}(GM(\mathbf{A}))) & \xrightarrow[\sim]{} & H^0(Sh(n,n)_{GP}, \mathcal{E}_{\mu,\kappa,GP}) \end{array} \tag{2.4.6}$$

Here $\lambda_{\mu,\kappa,n}^{-1}$ is the inverse of the character of $GP(\mathbb{R})$ which defines the line bundle $\mathcal{E}_{\mu,\kappa,GP}$, and $\mathcal{A}(GM(\mathbf{A}))$ is the space of automorphic forms on $GM(\mathbf{A})$. Strong approximation for $SL(n)$ implies that any irreducible automorphic representation of $GM(\mathbf{A})$ whose archimedean component is a character, in this case $\lambda_{\mu,\kappa,n}^{-1}$, is necessarily one-dimensional. The effect of $I_{GP(\mathbf{A}_f)}^{GH(\mathbf{A}_f)}$ is to induce at finite places but leave the archimedean place alone, cf. [13, Corollary 3.2.9]. The left-hand vertical arrow r_P is defined by the usual constant term map on $\mathcal{A}(n,n)$. The lower horizontal isomorphism also depends on the choice of a basis (canonical trivialization) of $\mathcal{E}_{\mu,\kappa,GP}$, which is determined by the remaining three arrows. As above, the right half of the diagram determines rational structures on the left half which makes r_P $E(\mu,\kappa)$-rational.

2.5 Twisting by characters on the Shimura variety

The map $\det : GH \to T_{\mathcal{K}} \overset{def}{\to} = R_{\mathcal{K}/\mathbb{Q}}(\mathbb{G}_m)_{\mathcal{K}}$ defines a morphism of Shimura data

$$(GH, X^d_{n,n}) \to (T_{\mathcal{K}}, h_{n,n}). \tag{2.5.1}$$

The homomorphism $h_{n,n} : R_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_m)_{\mathbb{C}} \to T_{\mathcal{K},\mathbb{R}}$ is defined by (2.5.1) and is given explicitly by the condition that, for every embedding $\sigma : \mathcal{K} \to \mathbb{C}$, the composition $\sigma \circ h_{n,n}(z) = (z\bar{z})^n$ (cf. [6, (2.9.1)]).

In particular, the image of $h_{n,n}$ is contained in the subtorus $T_E \overset{def}{\to} \to R_{E/\mathbb{Q}}(\mathbb{G}_m)_E$ of $T_{\mathcal{K}}$. By the yoga of automorphic vector bundles, this means that, if ρ is an algebraic character of $T_{\mathcal{K}}$ trivial on T_E, the corresponding automorphic line bundle $\mathcal{L}_\rho$ on $Sh(T_{\mathcal{K}}, h_{n,n})$ is $T_{\mathcal{K}}(\mathbf{A}_f)$-equivariantly isomorphic to the trivial line bundle. Thus, let β be a Hecke character of $\mathcal{K}$ that defines a section $[\beta] \in H^0(Sh(T_{\mathcal{K}}, h_{n,n}), \mathcal{L}_\rho)$; equivalently, $\beta_\infty = \rho^{-1}$ (cf. [6, (2.9.2)]). Then $[\beta]$ is a motivic Hecke character (character of type A_0), so the field $E[\beta]$ generated by the values of β on finite idèles is a finite extension of $\mathbb{Q}$.

Lemma 2.5.2. *Let E_ρ be the field of definition of the character ρ. The section $[\beta]$ is rational over the field $E[\beta]$; moreover, for any $\sigma \in Gal(\overline{\mathbb{Q}}/E_\rho)$, $\sigma[\beta] = [\sigma(\beta)]$, where σ acts on β by acting on its values on finite idèles.*

Let $\mathcal{E}_\rho$ denote the pullback of $\mathcal{L}_\rho$ to $Sh(n,n)$. Then twisting by $[\beta]$ defines an $E[\beta]$-rational isomorphism

$$i_\beta : H^0(Sh(n,n), \mathcal{E}_{\mu,\kappa}) \xrightarrow{\sim} H^0(Sh(n,n), \mathcal{E}_{\mu,\kappa} \otimes \mathcal{E}_\rho).$$

Lemma 2.5.2 implies that $\sigma \circ i_\beta = i_{\sigma(\beta)}$, where the notation has the obvious interpretation.

Complex conjugation c defines an automorphism, also denoted c, of the torus $T_{\mathcal{K}}$. Every integer k defines a character ρ_k of $T_{\mathcal{K}}$ trivial on T_E:

$$\rho_k(z) = (z/c(z))^{-k}.$$

Let $\alpha_k = \rho_k \otimes N^{-k}_{\mathcal{K}/\mathbb{Q}}$. One can also twist by sections $[\beta]$ of $H^0(Sh(T_{\mathcal{K}}, h_{n,n}), \mathcal{L}_{\alpha_k})$. The function β no longer defines a rational section, but we have:

Corollary 2.5.3. *Suppose β is a Hecke character with $\beta_\infty = \alpha_k^{-1}$, and let $[\beta] \in H^0(Sh(T_{\mathcal{K}}, h_{n,n}), \mathcal{L}_{\alpha_k})$ be the section defined by $(2\pi i)^{nk} \cdot \beta$. Then multiplication by $[\beta]$ defines an $E[\beta]$-rational isomorphism*

$$i_\beta : H^0(Sh(n,n), \mathcal{E}_{\mu,\kappa}) \xrightarrow{\sim} H^0(Sh(n,n), \mathcal{E}_{\mu,\kappa} \otimes \mathcal{E}_{\alpha_k}).$$

Moreover, for any $\sigma \in Gal(\overline{\mathbb{Q}}/E_\rho)$, $\sigma \circ i_\beta = i_{\sigma(\beta)}$.

2.6 The subvariety $Sh(n,n)^+ \subset Sh(n,n)$

Let $\underline{S}$ denote the real algebraic group $R_{\mathbb{C}/\mathbb{R}}\mathbb{G}_{m,\mathbb{C}}$. In Deligne's formalism, the space $X_{n,n}$ is a conjugacy class of homomorphisms $h : \underline{S} \to GH_{\mathbb{R}}$. The group $GK(n,n)$, introduced in Section 1, is the centralizer of one such h, namely $h_0 = (h_{0,\sigma})$, with $h_{0,\sigma}$, the projection of h on the factor $GU(n,n)_\sigma$ of $GH(\mathbb{R})$, given on $z = x + iy \in \underline{S}(\mathbb{R}) \simeq \mathbb{C}^\times$ by

$$h_{0,\sigma}(x+iy) = \begin{pmatrix} xI_n & yI_n \\ -yI_n & xI_n \end{pmatrix} \tag{2.6.1}$$

in the block matrix form of (1.1). Over $\mathbb{C}$ there is an isomorphism

$$(\mu, \mu') : \mathbb{G}_m \times \mathbb{G}_m \xrightarrow{\sim} \underline{S}$$

and the Cayley transform conjugates $h_{0,\mathbb{C}}$ to

$$r_0 = (r_{0,\sigma}) : \mathbb{G}_m \times \mathbb{G}_m \to \prod_\sigma GU(n,n)_\sigma,$$

where

$$r_{0,\sigma}(t,t') = \begin{pmatrix} tI_n & 0_n \\ 0_n & t'I_n \end{pmatrix}. \tag{2.6.2}$$

The cocharacter $\mu_{h_0} = h_{0,\mathbb{C}} \circ \mu : \mathbb{G}_m \to GH_{\mathbb{C}}$ is thus conjugate to the cocharacter μ_{r_0}, defined by

$$t \mapsto \begin{pmatrix} tI_n & 0_n \\ 0_n & I_n \end{pmatrix}. \tag{2.6.3}$$

The reflex field $E(GH, X^d_{n,n})$ is the field of definition of the conjugacy class of μ_{h_0}, or equivalently of μ_{r_0}; since μ_{r_0} is defined over $\mathbb{Q}$, it follows that $E(GH, X^d_{n,n}) = \mathbb{Q}$.

In particular, Shimura's reciprocity law for the connected components of $Sh(n,n)$ shows that these are defined over $\mathbb{Q}^{ab}$. We recall Shimura's reciprocity law in the version of Deligne [1, §2.6], which is correct up to a sign (depending on normalizations) that does not matter. Define a $\mathbb{Q}$-subgroup of $T_{\mathcal{K}} \times \mathbb{G}_m$ by

$$T = \{(u,t) \in T_{\mathcal{K}} \times \mathbb{G}_m \mid N_{\mathcal{K}/E}(t^{-n}u) = 1\}. \tag{2.6.4}$$

The map $d = \det \times \nu : GH \to T_{\mathcal{K}} \times \mathbb{G}_m$ takes values in T, and the simply-connected semisimple group $\ker d = SU(n,n)$ is the derived subgroup GH^{der} of GH. It follows from strong approximation for simply-connected semisimple groups, as in [1, §2.1], that the set $\pi_0(Sh(n,n))$ of geometrically connected components of $Sh(n,n)$ is a principal homogenous space under

$$\bar{\pi}_0\pi(G) = d(GH(\mathbf{A}_f))/\overline{d(GH(\mathbb{Q})_+)}.$$

Here $GH(\mathbb{Q})_+ = GH(\mathbb{Q}) \cap GH(\mathbb{R})^+$ and the bar over $d(GH(\mathbb{Q})_+)$ denotes topological closure.

Let $GH(\mathbf{A})^+ = \ker \varepsilon_{\mathcal{K}/E} \circ \nu$ as in (0.4), $GH(\mathbf{A}_f)^+ = GH(\mathbf{A})^+ \cap GH(\mathbf{A}_f)$, and let $Sh(n,n)^+$ denote the image of $(X^+_{n,n})^d \times GH(\mathbf{A}_f)^+$ in $Sh(n,n)$. This is a union of connected components of $Sh(n,n)$, defined over the subfield of $\mathbb{Q}^{ab}$ determined by the reciprocity law [1, (2.6.3)]. In [1], the reflex field E is just $\mathbb{Q}$, and for the map $q_M : \pi(\mathbb{G}_m) \to \pi(G)$, in Deligne's notation, we can just take the map μ_{r_0} of (2.6.3). The reciprocity map [1, (2.6.2.1)] is just deduced from the composite

$$d_{r_0} = d \circ \mu_{r_0} : \mathbb{G}_m \to T; t \mapsto (t^n, t). \tag{2.6.5}$$

It follows from the reciprocity law that $Sh(n,n)^+$ is defined over the field $L_{\mathcal{K}/E}$ defined by the kernel in $\mathbf{A}^\times/\mathbb{Q}^\times \cdot \mathbb{R}^\times_+ \xrightarrow{\sim} Gal(\mathbb{Q}^{ab}/\mathbb{Q})$ of

$$\varepsilon(n,n) = \varepsilon_{\mathcal{K}/E} \circ \nu \circ d_{r_0}. \tag{2.6.6}$$

This is a quadratic character, so $[L : \mathbb{Q}] \leq 2$. If $E = \mathbb{Q}$, it is easy to see that $L_{\mathcal{K}/E} = \mathcal{K}$; if $\mathcal{K}$ contains no quadratic extension of $\mathbb{Q}$, then $L_{\mathcal{K}/E} = \mathbb{Q}$.

The theory of automorphic vector bundles is valid over $Sh(n,n)^+$, provided $L_{\mathcal{K}/E}$ is taken as the base field. The Siegel–Weil formula only determines Eisenstein series over $GH(\mathbf{A})^+$, hence only determines the corresponding section of automorphic vector bundles on $Sh(n,n)^+$. Let $Sh(n,n)^+_{GP}$ be the point boundary stratum of the minimal compactification of $Sh(n,n)^+$; i.e., it is the intersection of $Sh(n,n)_{GP}$ with the closure of $Sh(n,n)^+$ in the minimal compactification. Then $Sh(n,n)^+_{GP}$ is also defined over $L_{\mathcal{K}/E}$, and Lemma 2.4.5 remains true with the superscript $^+$. We note the following:

Lemma 2.6.7. *Let $\mathcal{E}$ be any automorphic vector bundle over $Sh(n,n)$, $f \in H^0(Sh(n,n), \mathcal{E})$, and define $f^+ \in H^0(Sh(n,n), \mathcal{E})$ to equal f on $Sh(n,n)^+$ and zero on the complement. If $\mathcal{E}$ and f are rational over the field L, then f^+ is rational over $L_{\mathcal{K}/E} \cdot L$.*

Proof. Let 1 denote the constant section of $H^0(Sh(n,n), \mathcal{O}_{Sh(n,n)})$ identically equal to 1. Since $f^+ = f \cdot 1^+$, it suffices to verify the lemma for $f = 1$, but this follows immediately from the reciprocity law. □

3 The Siegel–Weil formula and arithmetic Eisenstein series

3.1 The case of rank one

The constant term of the theta lift is easy to calculate. We first consider the theta lift from $U(V)$ with $\dim V = 1$, as above. The standard calculation of the constant term (cf. [19, (1.3)]) gives

$$r_P(I_{\chi,triv,\psi}(\Phi))(h) = \tau(U(V)) \cdot \phi_\Phi(h), \; h \in GH^+(\mathbf{A}), \tag{3.1.1}$$

in the notation of (2.3.5) and (2.3.6). The constant $\tau(U(V))$ is the Tamagawa measure, equal to 2; all that matters to us is that it is a non-zero rational number. The map r_P is equivariant with respect to the action of $GH(\mathbf{A}_f)$, and we can extend the left-hand side to a function of $GH(\mathbf{A})$ so that (3.1.1) remains valid for all $h \in GH(\mathbf{A}_f)$.

Write $GH_\infty = GH(E \times_{\mathbb{Q}} \mathbb{R})$, and define $GU(V)_\infty$, $U(V)_\infty$, and H_∞ likewise. Howe duality for the pair $(H, U(V))$ implies that $\mathcal{S}(V^n)_\infty = \mathcal{S}(V^n)(E \times_{\mathbb{Q}} \mathbb{R})$ decomposes, as representation of $(U(Lie(H_\infty)), K(n,n))) \times U(Lie(U(V)_\infty)$, as an infinite direct sum of irreducible representations, indexed by a certain subset of the set of characters of the torus $U(V)_\infty$. Assuming $V_v = V^+$ and $\psi_v(x) = e^{a_v x}$ with $a_v > 0$ for all real places v of E, each summand is a holomorphic representation of H_∞. The summand corresponding to the trivial character of $U(V)_\infty$ is just the tensor product of d copies of $\mathbb{D}(\mu,\kappa)$, indexed by real places of E. Let $\Phi^0_\infty \in \mathcal{S}(V^n)_\infty$ be a non-zero $U(V)_\infty$-invariant function in the (one-dimensional) holomorphic subspace of $\mathbb{D}(\mu,\kappa)^{\otimes^d}$. With this choice of Φ^0_∞, unique up to scalar multiples, $I_{\chi,triv,\psi}(\Phi) \otimes |\nu^{\frac{n\kappa}{2}}|$ defines an element of $Hom_{(Lie(GH),GK(n,n))}(\tilde{\mathbb{D}}(\mu,\kappa), \mathcal{A}(n,n))$, and thus of $H^0(Sh(n,n), \mathcal{E}_{\mu,\kappa})$. The function Φ^0_∞ can be written explicitly as a Gaussian, but any choice will do.

Recall that (2.4.6) has been normalized so that the map (3.1.1) is rational over $E(\kappa,\mu)$. One can characterize the rational structure on the functions on the right-hand side of (3.1.1) explicitly, and some choices of Φ^0_∞ are more natural than others for this purpose, but this is unnecessary. The following proposition is an immediate consequence of Lemma 2.4.5.

Proposition 3.1.2. *Let L be an extension of $E(\kappa,\mu)$. Let $\Phi = \Phi^0_\infty \otimes \Phi_f \in \mathcal{S}(V(\mathbf{A})^n)$ be any function such that*

$$\phi_\Phi \otimes |\nu^{\frac{n\kappa}{2}}| \in H^0(Sh(n,n)_{GP}, \mathcal{E}_{\mu,\kappa,GP})(L) \tag{3.1.3}$$

(i.e., is rational over L) in terms of the bottom isomorphism in (2.4.6). Then $I_{\chi,triv,\psi}(\Phi) \otimes |\nu^{\frac{n\kappa}{2}}|$ defines an L-rational element of $H^0(Sh(n,n), \mathcal{E}_{\mu,\kappa})$ in terms of the top isomorphism in (2.4.6).

More generally, let $\gamma \in Gal(\overline{\mathbb{Q}}/E(\kappa,\mu))$ and suppose

$$\Phi_1 = \Phi^0_\infty \otimes \Phi_{1,f}, \Phi_2 = \Phi^0_\infty \otimes \Phi_{2,f} \in \mathcal{S}(V(\mathbf{A})^n)$$

have the property that ϕ_{Φ_i} is an L-rational element of $H^0(Sh(n,n)_{GP}, \mathcal{E}_{\mu,\kappa,GP})$, $i = 1,2$, for some $L \subset \overline{\mathbb{Q}}$ containing $E(\kappa,\mu)$, and such that $\gamma(\phi_{\Phi_1}) = \phi_{\Phi_2}$. Then

$$\gamma(I_{\chi,triv,\psi}(\Phi_1)) = I_{\chi,triv,\psi}(\Phi_2).$$

Remark. This can be improved by taking $\Phi_{i,f}$ to be $\overline{\mathbb{Q}}$-valued functions. The twist by a power of $|\nu|$ implicitly introduces a power of $(2\pi i)$ in the normalization, as in Corollary 2.5.3. This can be absorbed into Φ^0_∞ and is invisible in the above statement, because it is present in the boundary value as well as in the theta integral. However, the Galois group acts on the additive character ψ as well as on the values of $\Phi_{i,f}$, so the result is not completely straightforward. This is explained in some detail in Section (E.2) of [8]. Note that the statement of Proposition 3.1.2 depends only on the image of Φ_f in $R_n(V,\chi)_f$.

3.2 Siegel–Weil theta series in general

Now let V be of arbitrary dimension m, but always assume V to be positive-definite at all real places of E. Let A_V be the matrix of the Hermitian form of V, in some basis. Recall that H was defined to be $U(2W)$. For the theta correspondence it is best to view $2W$ as the skew-Hermitian space $\mathcal{W} = \mathcal{K}^{2n}$, with skew-Hermitian matrix $\begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$. Define $\mathbb{W} = \mathcal{W} \otimes V$. This is naturally a skew-Hermitian space, with matrix $\begin{pmatrix} 0 & I_n \otimes A_V \\ -I_n \otimes A_V & 0 \end{pmatrix}$. However, all even maximally isotropic skew-Hermitian spaces are isomorphic, so $\mathbb{W}$ is equivalent to the space with matrix $\begin{pmatrix} 0 & I_{nm} \\ -I_{nm} & 0 \end{pmatrix}$.

The subgroups H and $U(V)$ of $U(\mathbb{W})$ form a dual reductive pair. In particular, any automorphic form on $U(\mathbb{W})$ restricts to an automorphic form on $H \times U(V)$ whose integral over $U(V)(E)\backslash U(V)(\mathbf{A}_E)$ defines an automorphic form on H. More generally, the integral over $U(V)(E)\backslash U(V)(\mathbf{A}_E)$ of an automorphic form on $GU(\mathbb{W})$ defines an automorphic form on GH.

Let V_1 be the one-dimensional Hermitian space $\mathcal{K}$ with the norm form. We fix $\mu = \frac{1-\kappa}{2}$, as is appropriate for the theta lift from $U(V_1)$ to $U(\mathbb{W})$. Attached to $GU(\mathbb{W})$ we have the Shimura variety $Sh(nm, nm)$. We let $(GU(V), \{point\})$ be the trivial Shimura datum attached to $GU(V)$: $\{point\}$ is the conjugacy class of the trivial homomorphism $\mathbb{C}^\times \to GU(V)(E \times_{\mathbb{Q}} \mathbb{R})$ over $\mathbb{R}$. Let $Sh(V)$ be the corresponding (zero-dimensional) Shimura variety. Tensor product defines a natural homomorphism $GU(n,n) \times GU(V) \to GU(\mathbb{W})$ which induces a natural map of Shimura data:

$$(GU(n,n) \times GU(V), X_{n,n} \times \{point\}) \to (GU(\mathbb{W}), X_{nm,nm})$$

and hence a morphism of Shimura varieties

$$Sh(n,n) \times Sh(V) \to Sh(nm, nm) \tag{3.2.1}$$

defined over the reflex field, which is $\mathbb{Q}$. The pullback of $\mathcal{E}_{\mu,\kappa}$ from $Sh(nm,nm)$ to $Sh(n,n) \times Sh(V)$ is just the pullback from $Sh(n,n)$ of $\mathcal{E}_{m\mu,m\kappa}$. This is because the restriction to $K(n,n) = U(n) \times U(n)$ of the determinant character

$$K(nm,nm) = U(nm) \times U(nm) \to U(1) \times U(1),$$

defined by (two copies of) the diagonal homomorphism

$$U(n) \to U(nm) = U(V^n),$$

is the m-power of the determinant character on $K(n,n)$.

Lemma 3.2.2. *Let* $\tau = \tau(U(V)) = 2$. *There is a commutative diagram*

$$\begin{array}{ccc} Hom_{(GU(\mathbb{W}),GK(nm,nm))}(\tilde{\mathbb{D}}(\mu,\kappa),\mathcal{A}(nm,nm)) & \xrightarrow[\sim]{} & H^0(Sh(nm,nm),\mathcal{E}_{\mu,\kappa}) \\ \Big\downarrow \tau(U(V))^{-1}\int_{U(V)(E)\backslash U(V)(\mathbf{A}_E)} & & I_V \Big\downarrow \\ Hom_{(Lie(GH),GK(n,n))}(\tilde{\mathbb{D}}(m\mu,m\kappa),\mathcal{A}(n,n)) & \xrightarrow[\sim]{} & H^0(Sh(n,n),\mathcal{E}_{m\mu,m\kappa}) \end{array}$$

where the right vertical arrow is $E(\mu,\kappa) = E(m\mu,m\kappa)$*-rational.*

Proof. The fibers of $\mathcal{E}_{\mu,\kappa}$ on $Sh(nm,nm)$ and of $\mathcal{E}_{m\mu,m\kappa}$ on a point in the image of $Sh(n,n)$ in $Sh(nm,nm)$ are identical, so the trivialization (horizontal) maps can be made compatible. The right hand arrow I_V is then just given by projection on $U(V)(\mathbf{A}_f)$-invariants – which form a direct summand, since $U(V)$ is anisotropic – followed by restriction to $Sh(n,n)$. The action of $GU(\mathbb{W})(\mathbf{A}_f)$ is $E(\mu,\kappa)$-rational, so every step in the above description of the right-hand arrow is $E(\mu,\kappa)$-rational. □

We let P_{nm} and P_n denote the Siegel parabolic subgroups of $U(\mathbb{W})$ and H, respectively, and define GP_{nm} and GP_n likewise.

Lemma 3.2.3. *There is a commutative diagram*

$$\begin{array}{ccc} H^0(Sh(nm,nm),\mathcal{E}_{\mu,\kappa}) & \xrightarrow{\mathfrak{f}_{P_{nm}}} & H^0(Sh(nm,nm)_{GP_{nm}},\mathcal{E}_{\mu,\kappa,GP_{nm}}) \\ \Big\downarrow I_V & & \Big\downarrow \\ H^0(Sh(n,n),\mathcal{E}_{m\mu,m\kappa}) & \xrightarrow{\mathfrak{f}_{P_n}} & H^0(Sh(n,n)_{GP_n},\mathcal{E}_{m\mu,m\kappa,GP_n}). \end{array}$$

The right-hand vertical arrow is given by restriction of functions from $GU(\mathbb{W})(\mathbf{A}_f)$ *to* $GH(\mathbf{A}_f)$.

Proof. The two dual reductive pairs $(U(2W),U(V))$ and $(U(\mathbb{W}),U(V_1))$ form a seesaw:

$$\begin{array}{cc} U(\mathbb{W}) & U(V) \\ | & | \\ U(2W) & U(V_1). \end{array}$$

Here $U(V_1)$ is just the center of $U(V)$. The splittings of the metaplectic cover for the two pairs are compatible if one takes the character χ for the pair $(U(\mathbb{W}),U(V_1))$, with $\chi\mid_{\mathbf{A}_E^\times} = \varepsilon$, and $\chi^{\dim V} = \chi^m$ for the pair $(U(2W),U(V))$ (cf. [4, §3] for the rules governing compatible splittings. □

Corollary 3.2.4. *Let* χ *be a Hecke character of* $\mathcal{K}^\times$ *satisfying* $\chi\mid_{\mathbf{A}_E^\times} = \varepsilon$. *Let* L *be an extension of* $E(\kappa,\mu)$. *Let* $\Phi = \Phi_\infty^0 \otimes \Phi_f \in \mathcal{S}(V(\mathbf{A})^n)$ *be any function such that* $\phi_\Phi \otimes |\nu^{\frac{nm\kappa}{2}}|$ *is an* L*-rational element of* $H^0(Sh(n,n)_{GP}, \mathcal{E}_{m\mu,m\kappa,GP})$ *in terms of the bottom isomorphism in (2.4.6). Then* $I_{\chi^m,triv,\psi}(\Phi)\otimes|\nu^{\frac{nm\kappa}{2}}|$ *defines an* L*-rational element of* $H^0(Sh(n,n),\mathcal{E}_{m\mu,m\kappa})$ *in terms of the top isomorphism in (2.4.6).*

More generally, let $\gamma \in Gal(\overline{\mathbb{Q}}/E(\kappa,\mu))$ *and suppose*

$$\Phi_1 = \Phi_\infty^0 \otimes \Phi_{1,f},\ \Phi_2 = \Phi_\infty^0 \otimes \Phi_{2,f} \in \mathcal{S}(V(\mathbf{A})^n)$$

are such that ϕ_{Φ_i} *is an* L*-rational element of* $H^0(Sh(n,n)_{GP},\mathcal{E}_{m\mu,m\kappa,GP})$, $i=1,2$, *for some* $L\subset\overline{\mathbb{Q}}$ *containing* $E(m\kappa,m\mu)$, *and such that* $\gamma(\phi_{\Phi_1}) = \phi_{\Phi_2}$. *Then*

$$\gamma(I_{\chi^m,triv,\psi}(\Phi_1)) = I_{\chi^m,triv,\psi}(\Phi_2).$$

Proof. We consider the first claim, the proof of the second claim being similar. When n is replaced by nm, V by V_1, and $(m\mu,m\kappa)$ by (μ,κ), this is just Proposition 3.1.2. Now by construction $\mathbb{W}\otimes V_1 = 2W\otimes V$ so $V_1(\mathbf{A})^{nm}$ is isomorphic to $V(\mathbf{A})^n$. We apply Proposition 3.1.2 to the top line in the commutative diagram in Lemma 3.2.3 and obtain the conclusion under the stronger hypothesis that ϕ_Φ is an L-rational element of $H^0(Sh(nm,nm)_{GP},\mathcal{E}_{\mu,\kappa,GP})$.

Now consider the commutative diagram

$$\begin{array}{ccc} \mathcal{S}(V_1(\mathbf{A}_f)^{nm}) & \xrightarrow{\phi_{\bullet,nm}} & H^0(Sh(nm,nm)_{GP_{nm}},\mathcal{E}_{\mu,\kappa,GP_{nm}}) \\ \downarrow = & & \downarrow \\ \mathcal{S}(V(\mathbf{A}_f)^n) & \xrightarrow{\phi_{\bullet,nm}} & H^0(Sh(n,n)_{GP_n},\mathcal{E}_{m\mu,m\kappa,GP_n}). \end{array} \tag{3.2.5}$$

Here $\phi_{\bullet,nm}$, resp. $\phi_{\bullet,n}$, is the map $\Phi\mapsto\phi_\Phi$ for the pair $(U(\mathbb{W}),U(1))$, resp. $(U(2n),U(V))$. By (3.1.1), we have

$$\phi_{\bullet,nm} = 2\cdot r_P\cdot I_{\chi,triv,\psi};\quad \phi_{\bullet,n} = 2\cdot r_P\cdot I_{\chi^m,triv,\psi}.$$

Let $B_{nm} = Im(\phi_{\bullet,nm})$, $B_n = Im(\phi_{\bullet,n})$. It follows from Corollary 2.4.3 that B_{nm} is an $E(\mu,\kappa)$-rational subspace of $H^0(Sh(nm,nm)_{GP_{nm}},\mathcal{E}_{\mu,\kappa,GP_{nm}})$. But the equality on the left of (3.2.5) implies that B_n is the image of B_{nm} under the right-hand vertical map. Thus, B_n is also $E(\mu,\kappa)$-rational and any L-rational vector in B_n can be lifted to an L-rational vector in B_{nm}. It thus follows that, with $I_{\chi^m,triv,\psi}(\Phi)$ as in the statement of the lemma, we can assume $\phi_{\Phi,nm}$ as well as $\phi_{\Phi,n}$ is L-rational. The corollary then follows from Proposition 3.1.2 and Lemma 3.2.3. □

Corollary 3.2.4 is stated in terms of the character χ, but could just as well be stated in terms of the character χ^m; the condition is that the splitting character used to define the theta lift has to be an m-th power, where $m = \dim V$. However, this is unnecessary:

Corollary 3.2.6. *The assertions of Corollary 3.2.4 hold when* $I_{\chi^m,triv,\psi}(\Phi)$ *is replaced by* $I_{\xi,triv,\psi}(\Phi)$ *where* ξ *is any Hecke character of* $\mathcal{K}^\times$ *satisfying* $\xi\,|_{\mathbf{A}_E^\times} = \varepsilon^m$.

Proof. Indeed, by Theorem 2.1.5(c),

$$I_{\xi,triv,\psi}(\Phi) = I_{\chi^m,triv,\psi}(\Phi) \otimes (\xi/\chi^m) \circ \det .$$

Let $\beta = \xi/\chi^m$. This is a character trivial on the idèles of E, hence of the type considered in Section 2.5. Bearing in mind the various implicit normalizations and character twists, Corollary 3.2.6 follows from Corollaries 2.5.3 and 3.2.4. □

3.3 Applications of the Siegel–Weil formula

We now assume V is a positive-definite Hermitian space over $\mathcal{K}$ of dimension $m \geq n$. Let $s_0 = \frac{m-n}{2}$. The Eisenstein series is normalized as in Section 1. Let $\Phi \in \mathcal{S}(V^n)(\mathbf{A})$, and let χ satisfy (1.2.4). Ichino has proved the following analogue of results of Kudla and Rallis:

Theorem 3.3.1. *[16] The extended Siegel–Weil formula is valid for* Φ*: the Eisenstein series* $E(h, s, \phi_\Phi, \chi)$ *has no pole at* $s = s_0$*, and*

$$I_{\chi,triv,\psi}(\Phi)(h) = c \cdot E(h, s_0, \phi_\Phi, \chi)$$

for $h \in H(\mathbf{A})$*, where* $c = 1$ *if* $m = n$ *and* $c = \frac{1}{2}$ *otherwise.*

For applications to special values, we need the analogue of Ichino's theorem for $h \in GH(\mathbf{A})^+$.

Corollary 3.3.2. *The extended Siegel–Weil formula is valid on* $GH(\mathbf{A})^+$:

$$I_{\chi,triv,\psi}(\Phi)(h) = c \cdot E(h, s_0, \phi_\Phi, \chi)$$

for $h \in GH(\mathbf{A})^+$*, where* $c = 1$ *if* $m = n$ *and* $c = \frac{1}{2}$ *otherwise.*

Proof. Since both sides are left-invariant under $GH(\mathbb{Q})$, it suffices to establish the identity for $h \in GH^+(\mathbf{A})$. The extension from $H(\mathbf{A})$ to $GH^+(\mathbf{A})$ is carried out as in Section 4 of [10]. □

Recall the number field $L_{\mathcal{K}/E}$ defined by the character (2.6.6). Corollary 3.2.6 then immediately has the following consequence:

Corollary 3.3.3. *Let* χ *be a Hecke character of* $\mathcal{K}^\times$ *satisfying* $\chi\,|_{\mathbf{A}_E^\times} = \varepsilon$. *Let* L *be an algebraic extension of* $L_{\mathcal{K}/E} \cdot E(\kappa, \mu)$. *Let* $\Phi = \Phi^0_\infty \otimes \Phi_f \in \mathcal{S}(V(\mathbf{A})^n)$ *be any function such that* $f = \phi_\Phi$ *is an* L*-rational element of* $H^0(Sh(n,n)^+_{GP}, \mathcal{E}_{m\mu,\kappa,GP})$ *in terms of the bottom isomorphism in (2.4.6).*

Then $E(h, s_0, f, \chi)$ *defines an* L*-rational element of* $H^0(Sh(n,n)^+, \mathcal{E}_{m\mu,m\kappa})$ *in terms of the top isomorphism in (2.4.6).*

More generally, if $f \in H^0(Sh(n,n)^+_{GP}, \mathcal{E}_{m\mu,\kappa,GP})$ *is an* L*-rational Siegel–Weil section for the pair* $(U(2W), U(V))$*, then for all* $\gamma \in Gal(\overline{\mathbb{Q}}/L_{\mathcal{K}/E} \cdot E(\mu,\kappa))$

$$\gamma(E(h, s_0, f, \chi)) = E(h, s_0, \gamma(f), \gamma(\chi))$$

where γ *acts on the finite part of* χ*.*

Earlier work of Ichino [15] considered the case of $m < n$. The results of [15] are valid whether or not V is a definite Hermitian space, and identify certain residues of Eisenstein series with explicit theta functions. In this way one can apply Corollary 3.2.6 to residues in certain cases. Since this is unnecessary for applications to special values of L-functions, we omit the details.

4 Special values of L-functions

We provide the anticipated application to special values of L-functions, extending Theorem 3.5.13 of [6] to the middle of the critical strip for characters α satisfying (1.2.6). As explained in the introduction, this is a somewhat restrictive hypothesis, about which more will be said later. For the sake of simplicity, we assume $E = \mathbb{Q}$, so that $\mathcal{K}$ is imaginary quadratic, and the field $L_{\mathcal{K}/E} = \mathcal{K}$. The techniques can be applied without much difficulty to general CM fields.

Recall that in [6], whose notation we use without further explanation, we have chosen a cuspidal automorphic representation π of $G = GU(W)$ with $\dim W = n$, $G(\mathbb{R}) \xrightarrow{\sim} GU(r,s)$, and two algebraic Hecke characters χ and α of fixed weights. It is assumed that π, or rather its finite part π_f, occurs non-trivially in the middle-dimensional cohomology $\bar{H}^{rs}(W_\mu^\nabla)$ of the Shimura variety naturally associated to G with coefficients in the local system denoted W_μ^∇ ($\bar{H}$ denotes the image of cohomology with compact support in cohomology). In [6] the following condition, which should hold automatically, was inadvertently omitted:

Condition 4.1. The representation π contributes to the antiholomorphic component of $\bar{H}^{rs}(W_\mu^\nabla)$.

The anti-holomorphy of π on G is equivalent to the holomorphy of π viewed as an automorphic representation of the isomorphic group $GU(-W)$, where $-W$ is the space W with its Hermitian form multiplied by -1; this amounts in the notation of [6] to replacing the Shimura datum $(G, X_{r,s})$ by the (complex conjugate) Shimura datum $(G, X_{s,r})$. In fact, this holomorphy (or anti-holomorphy) is the only property used in the proof of Theorem 3.5.13, and was assumed explicitly on page 151 of [6] but omitted in the statement of the theorem. The hypothesis of belonging to middle-dimensional cohomology is only made in keeping with the overall motivic theme of [6] and is in fact irrelevant to the proof, which works just as well for holomorphic forms contributing to cohomology in other degrees. Let $G(W,-W) \subset G \times G = GU(W) \times GU(-W)$ be the subgroup of pairs (g,g'), $g,g' \in G$, with equal similitude factors. We let $Sh(W) = Sh(G, X_{r,s})$, $Sh(-W) = Sh(G, X_{s,r})$, and let $Sh(W,-W)$ be the Shimura variety corresponding to $(G(W,-W), X_{r,s} \times X_{s,r})$.

Let μ be the highest weight of a finite-dimensional representation of G (i.e., of $G(\mathbb{C})$). Then μ can be represented, as in [6, 2.1], by an $(n+1)$-tuple

$$(a_1, \dots, a_r; a_{r+1}, \dots, a_n; c)$$

of integers with $c \equiv \sum_i a_j \pmod 2$ and $a_1 \geq a_2 \cdots \geq a_n$. To μ we can also associate an automorphic vector bundle E_μ on $Sh(W)$. If $K_\infty \subset G(\mathbb{R})$ is the stabilizer of a base point in $X_{r,s}$ – then K_∞ is a maximal connected subgroup of $G(\mathbb{R})$, compact modulo the center – then μ is also the highest weight of an irreducible representation of K_∞ and E_μ is obtained from the corresponding Hermitian equivariant vector bundle on $X_{r,s}$, cf. [6, 2.2]. If $k \in \mathbb{Z}$ we let $\eta_k(z) = z^{-k}$ for $z \in \mathbb{C}^\times$ and say the algebraic Hecke character χ of $\mathcal{K}^\times$ is *of type* η_k if $\chi_\infty = \eta_k$.

Let S be a finite set of finite places of $\mathbb{Q}$. In what follows, S will be any set containing all places at which π, χ, α, are unramified; however, see Remark (4.4)(v) for clarification. We define the motivically normalized standard L-function, with factors at S (and archimedean factors) removed, to be

$$L^{mot,S}(s, \pi \otimes \chi, St, \alpha) = L^S(s - \frac{n-1}{2}, \pi, St, \alpha). \tag{4.2}$$

The motivically normalized standard zeta integrals are defined by the corresponding shift in the integrals [6, (3.2.5)]. As in [6, (2.2.5) and (2.2.9)], define

$$p_j(\mu) = a_j + n - j + \mathcal{P}(\mu);\ \ q_j(\mu) = -a_j + j - 1 + \mathcal{Q}(\mu),$$

where

$$\mathcal{P}(\mu) = -\frac{1}{2}[c + \sum_{j=1}^n a_j];\ \ \mathcal{Q}(\mu) = -\frac{1}{2}[c - \sum_{j=1}^n a_j].$$

With these conventions, Theorem 3.5.13 of [6] is the special case of the following theorem in which $m > n - \frac{\kappa}{2}$. Unexplained notation is as in [6]:

Theorem 4.3. *Let $G = GU(W)$, a unitary group with signature (r,s) at infinity, and let π be a cuspidal automorphic representation of G. We assume $\pi \otimes \chi$ occurs in antiholomorphic cohomology $\bar{H}^{rs}(Sh(W), E_\mu)$ where μ is the highest weight of a finite-dimensional representation of G. Let χ and α be algebraic Hecke characters of $\mathcal{K}^\times$ of type η_k and η_κ^{-1}, respectively. Let s_0 be an integer which is critical for the L-function $L^{mot,S}(s, \pi \otimes \chi, St, \alpha)$; i.e., s_0 satisfies the inequalities (3.3.8.1) of [6]:*

$$\frac{n-\kappa}{2} \leq s_0 \leq min(q_{s+1}(\mu) + k - \kappa - \mathcal{Q}(\mu), p_s(\mu) - k - \mathcal{P}(\mu)), \tag{4.3.1}$$

where notation is as in [loc. cit]. Define $m = 2s_0 - \kappa$*. Let* α^* *denote the unitary character* $\alpha/|\alpha|$ *and assume*

$$\alpha^* \mid_{\mathbf{A}_{\mathbb{Q}}^\times} = \varepsilon_{\mathcal{K}}^m. \tag{4.3.2}$$

Suppose there is a positive-definite Hermitian space V *of dimension* m*, a factorizable section* $\phi_f(h, s, \alpha^*) \in I_n(s, \alpha^*)_f$*, and factorizable vectors* $\varphi \in \pi \otimes \chi$*,* $\varphi' \in \alpha^* \cdot (\pi \otimes \chi)^\vee$*, such that*

(a) For every finite v*,* $\phi_v \in R_n(V_v, \alpha^*)$*;*
(b) For every finite v*, the normalized local zeta integrals* $\tilde{Z}_v^{mot}(s, \varphi_v, \varphi'_v, \phi_v, \alpha_v^*)$ *do not vanish at* $s = s_0$*.*

Then

(i) One can find ϕ_f*,* φ*,* φ' *satisfying (a) and (b) such that* ϕ_f *takes values in* $(2\pi i)^{(s_0+\kappa)n} L \cdot \mathbb{Q}^{ab}$*, and such that* φ*,* φ' *are arithmetic over the field of definition* $E(\pi)$ *of* π_f*.*
(ii) Suppose φ *is as in (i). Then*

$$L^{mot,S}(s_0, \pi \otimes \chi, St, \alpha) \sim_{E(\pi,\chi^{(2)}\cdot\alpha);\mathcal{K}} P(s_0, k, \kappa, \pi, \varphi, \chi, \alpha)$$

where $P(s_0, k, \kappa, \pi, \varphi, \chi, \alpha)$ *is the period*

$$\pi^c (2\pi i)^{s_0 n - \frac{nw}{2} + k(r-s) + \kappa s} g(\varepsilon_{\mathcal{K}})^{[\frac{n}{2}]} \cdot P^{(s)}(\pi, *, \varphi) \cdot g(\alpha_0)^s \cdot p((\chi^{(2)} \cdot \alpha)^\vee, 1)^{r-s}$$

appearing in Theorem 3.5.13 of [6].

4.4 Remarks

(i) Condition (b) can be reinterpreted as the condition that the theta lift from $\pi_v \otimes \chi_v$ to $U(V)(E_v)$ be non-trivial, cf. [11, top of p. 975]. This will be discussed in more detail in [8]. There it will be shown that, as long as π is locally tempered at all v, as will generally be the case for the π of arithmetic interest, and as long as $m - n \geq 2$, one can always find V and functions ϕ_v satisfying (a) and (b). When $m = n$ this is still possible unless the L-function vanishes at s_0, in which case the conclusion is vacuously true. When $m = n + 1$ the conditions are necessary.
(ii) In Theorem 3.15.3 of [6] π_f was assumed tempered. The hypothesis is used at no point in the proof and was only included because of the motivic context. However, the hypothesis does allow certain simplifications, as already indicated in the previous remark. If one assumes π admits a (weak) base change to a cuspidal automorphic representation of $GL(n, \mathcal{K})$ as in [12, 3.1.3], then the local Euler factors at bad primes are the standard (Godement–Jacquet) local Euler factors of the base change; moreover, π_v is locally tempered everywhere [12, 3.1.5], hence these Euler factors have no poles to the right of the center of symmetry. In this case, or more generally if π_v is assumed tempered everywhere, one can replace "normalized zeta integrals" by "zeta integrals" in condition (b).

(iii) Garrett's calculation of the archimedean local factor, cited without proof in [6] as Lemma 3.5.3, has now appeared as Theorem 2.1 of [3]. Garrett's formulation is slightly different from that assumed in [6]; whereas the latter asserted that the archimedean local factor is an element of $\mathcal{K}^{\times}$, Garrett's integral is a $\mathcal{K}$-multiple of π^{rs}. Remark (iv) below explains why one can conclude that Garrett's integral does not vanish. The power of π in Garrett's integral is compensated by our choice of measure (see the formula on p. 83 of [6]).
There is a more subtle difference. Garrett calculates an integral over G as an operator on the representation π, whereas the archimedean zeta integral in [6] is an integral over G and gives a number as a result. One obtains a number by taking $g = 1$ in Garrett's Theorem 2.1. However, this process implicitly depends on the choice of base point (choice of maximal compact subgroup) and one needs to compare Garrett's implicit normalizations with those considered in [6]. This point is addressed in Section 5.

(iv) The meaning of the inequalities (4.3.1) is that the anti-holomorphic representation $[\pi \otimes \chi] \otimes [\pi^{\vee} \otimes (\chi \cdot \alpha)^{-1}]$ pairs non-trivially with the holomorphic subspace $I_n(s_0, \alpha^*)^{hol}$ of the degenerate principal series representation $I_n(s_0, \alpha^*)$; this is the content of Lemma 3.3.7 and Corollary 3.3.8 of [6]. It is well-known, and follows from the results of [22], particularly Proposition 5.8, that

$$I_n(s_0, \alpha^*)^{hol} = R_n(V(m, 0), \alpha^*) \tag{4.4.1}$$

where $V(m, 0)$ is the positive-definite Hermitian space over $\mathbb{C}$ of dimension m. As in [11], Proposition 3.1, the hypothesis that s_0 satisfies (4.3.1) then implies that (the contragredient of) $[\pi \otimes \chi] \otimes [\pi^{\vee} \otimes (\chi \cdot \alpha)^{-1}]$ has a non-trivial theta lift locally to $U(V(m, 0)) \times U(V(m, 0))$. By Lemma 2.3.13 of [8], it then follows that the (analytic continuation of the) archimedean local zeta integral calculated up to rational factors by Garrett does not vanish at $s = s_0$. Thus the archimedean counterpart of condition (b) is an automatic consequence of (4.3.1).
There is a global proof of the non-vanishing of the archimedean zeta integral that is simpler but requires more notation from [6], as well as some notation for totally real fields, which I prefer to leave to the reader's imagination. The holomorphic Eisenstein series denoted $E(g, \alpha, s, \phi)$ in Corollary 3.3.10 of [6] is non-zero at $s = 0$ (because its constant term is non-zero). Thus $\Delta(m, \kappa, \Lambda)E(g, \alpha, 0, \phi) \neq 0$ (because the archimedean component (4.4.1) is irreducible). Now if we replace the base field $\mathbb{Q}$ by a real quadratic extension L, with real places, and $\mathcal{K}$ by the CM field $\mathcal{K} \cdot L$ then the analogous fact remains true. Let σ_1, σ_2 be the real places of L and suppose W is a Hermitian space over $\mathcal{K}\dot{L}$ with signatures (r, s) at σ_1 and $(n, 0)$ at σ_2, and let $\Delta_{\sigma_1}(m, \kappa, \Lambda)$ be the differential operator defined locally as in [6] at the place σ_1. (Note that in [6] the Hermitian space is denoted V rather than W.) Then the holomorphic automorphic form $\Delta_{\sigma_1}(m, \kappa, \Lambda)E(g, \alpha, 0, \phi)$ on $Sh(W, -W)$ is still non-trivial. But $U(W)$ is

now anisotropic, so $\Delta_{\sigma_1}(m,\kappa,\Lambda)E(g,\alpha,0,\phi)$ is cuspidal. It follows that the analogue for $Sh(W)$ of the cup product map in Corollary 3.3.10 is non-trivial. In other words, the integral of $\Delta_{\sigma_1}(m,\kappa,\Lambda)E(g,\alpha,0,\phi)$ against some factorizable anti-holomorphic cusp form on $Sh(W,-W)$ of the right infinity type is non-zero. Thus all the local zeta integrals in the factorization [6, (3.2.4)] are non-vanishing at $s=s_0$. But the local integral at σ_1 is the one of interest to us.

(v) In [6] the partial L-function L^S was considered because I did not want to worry about the local factors at bad places. There are now two ways to define these factors: as least common denominators of zeta integrals with respect to "good sections," as in [11]; or by the method of Lapid and Rallis in [21]. If π admits a base change to an automorphic representation of $GL(n,\mathcal{K})$, then we have a third way to define the local Euler factors at S. The last two definitions are known to be compatible with functional equations including the same archimedean factors, hence are equal by a standard argument. Of course the missing local factors should be Euler factors with coefficients in the field $E(\pi,\chi^{(2)}\cdot\alpha)$, and one can then replace $L^{mot,S}$ by L^{mot} in the statement of Theorem 4.3. One can prove this directly for the factors defined in [11] and for the Euler factors of base change. I have not verified that there is a direct proof of rationality for the Euler factors of [21].

4.5 $GH(\mathbf{A})$ vs. $GH(\mathbf{A})^+$

The proof of Theorem 4.3 presented below follows the treatment of the absolutely convergent case in [6], with the difference that we have only proved arithmeticity of the Siegel–Weil Eisenstein series on the variety $Sh(n,n)^+$. This makes no difference to the final result, where the special value is only specified up to the equivalence relation $\sim_{E(\pi,\chi^{(2)}\cdot\alpha);\mathcal{K}}$, but we need to modify the argument in two points. The notation of [6] is used without comment.

4.5.1

The basic identity of Piatetski-Shapiro and Rallis is stated in [6, (3.2.4)] for unitary similitude groups. Let $G=G(U(V)\times U(-V))\subset GH$, as in [loc. cit.], and let $G(\mathbf{A})^+=G(\mathbf{A})\cap GH(\mathbf{A})^+$. Define the modified zeta integral

$$Z^+(s,f,f',\alpha,\phi)=\int_{[Z(\mathbf{A})\cdot G(\mathbb{Q})\backslash G(\mathbf{A})^+]^2} E(i_V(g,g'),\alpha,s,\phi)f(g)f'(g')dgdg',$$

where notation is as in [H3,(3.2.3)]. Then [6, (3.2.4)] is replaced by

$$d_n^S(\alpha,s)Z^+(s,f,f',\alpha,\phi)=(f^+,f')_{V,\alpha}Z_S L^{mot,S}(s+\frac{1}{2}n,\pi,St,\alpha), \quad (4.5.2)$$

where Z_S is the product of the local zeta integrals at places in S, and the automorphic form f^+ is defined as in Lemma 2.6 to equal f on $G(\mathbf{A})^+$ and zero on the complement. The point is that the Euler factors only see the unitary group, and the similitudes are incorporated into the automorphic period factor $(f^+, f')_{V,\alpha}$.

4.5.3

The motivic period factor $P^{(s)}(\pi, *, \varphi)$, also written $P^{(s)}(\pi, V, \varphi)$, is related to the automorphic period $(f, f')_{V,\alpha}$ by formulas (3.5.10), (3.5.8.2), and (3.5.12.1) of [6]:

$$(2\pi)^c (f, f')_{V,\alpha} \sim_{E(\pi,\chi^{(2)}\cdot\alpha);\mathcal{K}} P^{(s)}(\pi, V, \beta)^{-1} \cdot X(k, r, s, \chi, \alpha), \tag{4.5.4}$$

where $X(k, r, s, \chi, \alpha)$ is an explicit abelian period and $f = \beta \otimes \chi$, $f' = \beta' \otimes (\chi\alpha)^{-1}$ in the notation of [6, §3.5]. Given (4.5.2), it now suffices to show that

$$(f, f')_{V,\alpha} \sim_{E(\pi,\chi^{(2)}\cdot\alpha);\mathcal{K}} (f^+, f')_{V,\alpha} \tag{4.5.5}$$

when $f = \beta \otimes \chi$ and f' are chosen to correspond to arithmetic anti-holomorphic forms. But this is an easy consequence of Lemma 2.6.7 (and Serre duality, which translates Lemma 2.6.7 into a dual assertion concerning anti-holomorphic forms).

Proof (of Theorem 4.3). We first observe that the subspaces $R_n(V_v, \alpha_v^*) \subset I_n(s_0, \alpha_v^*)$, for v finite, are rational over the field of definition of the finite part α_f of α. Indeed, the induced representation $I_n(s_0, \alpha_v)$ has a natural model as a space of functions transforming with respect to a certain character of the maximal parabolic P. This model is visibly defined over the field of definition of α_v. Indeed, the modulus character (1.1.1) is in general only defined over $\mathbb{Q}(\sqrt{p_v})$, where p_v is the residue characteristic of v, but one verifies that the half-integral shift in the motivic normalization ensures that no odd powers of the square root of the norm occur for critical values of s.[3] It is obvious (by considering restriction to $GU(2W, \mathcal{O}_v)$, for instance) that $\sigma(R_n(V_v, \alpha_v)) = R_n(V_v, \sigma(\alpha_v))$ for any $\sigma \in Gal(\overline{\mathbb{Q}}/\mathcal{K})$. Finally, the pairing (2.1.7.1) – with π replaced by $\pi \otimes \chi$ – is defined over the field $E(\pi, \chi, \alpha) \cdot \mathbb{Q}^{ab}$. This completes the verification of (i).

With (i) in hand, (ii) is proved exactly as in the absolutely convergent case in [6], bearing in mind the modifications (4.5.1) and (4.5.3). First suppose $L^{mot,S}(s_0, \pi \otimes \chi, St, \alpha) \neq 0$. As already mentioned in [6, 3.8], in order to treat the general case of critical s_0 to the right of the center of symmetry, it suffices to show that the holomorphic Eisenstein series that enter into the proof remain arithmetic at the point corresponding to s_0, and that one can find Eisenstein series that pair non-trivially with arithmetic vectors in $\pi \otimes \alpha \cdot \pi^\vee$.

[3] In [6] the rationality of the induced representation at critical points is implicitly derived from the rationality of the boundary data lifted in the definition of the Eisenstein series.

Corollary 3.3.2 asserts that the holomorphic Eisenstein series are arithmetic provided they are attached to arithmetic sections ϕ_f that satisfy (a). Given the basic identity of Piatetski-Shapiro and Rallis [6, 3.2.4] the non-triviality of the pairing is guaranteed by (b), Remark 4.4 (iv), and our hypothesis on non-vanishing of the L-value, and the proof is complete.

On the other hand, if $L^{mot,S}(s_0, \pi \otimes \chi, St, \alpha) = 0$, we need to prove that $L^{mot,S}(s_0, \pi^\sigma \otimes \chi^\sigma, St, \alpha^\sigma) = 0$ for all $\sigma \in Gal(\overline{\mathbb{Q}}/\mathcal{K})$. But this is standard (cf. [29]): since we can choose arithmetic data for which for the local zeta integrals at primes dividing S do not vanish at s_0, it suffices to observe that the global pairing between Eisenstein series and (anti-holomorphic) cusp forms is rational over the reflex field of $Sh(W, -W)$, which is either $\mathbb{Q}$ or $\mathcal{K}$. □

5 Normalizations: Comparison with [3]

Garrett's calculation in [3] of archimedean zeta integrals, up to algebraic factors, is based on a choice of abstract rational structure on the enveloping algebra and its holomorphic highest weight modules. The zeta integrals considered in Section 4 and in [6, Lemma 3.5.3], involve explicit choices of data, especially an explicit choice of automorphy factor. The purpose of the present section is to verify that these two rational structures are compatible. In what follows $E = \mathbb{Q}$ and $\mathcal{K}$ is an imaginary quadratic field, but it should be routine to modify these remarks to apply to the general case.

The rationality invoked in [6] is that inherited from [4]. The group G, here $GU(W)$, is rational over $\mathbb{Q}$. The compact dual symmetric space $\hat{M} = \hat{M}(G, X)$ is endowed with a natural structure over the reflex field $E(G, X)$, which in our case is contained in $\mathcal{K}$. This natural $\mathcal{K}$-rational structure is compatible with the action of G, and all G-equivariant vector bundles on $\hat{M}$, along with the corresponding automorphic vector bundles on $Sh(G, X)$, are naturally defined over $\mathcal{K}$. All these remarks apply equally to the group $H = GU(2W)$. Though the corresponding Shimura variety, which we denote $Sh(H, X_{n,n})$ (cf. [6] for $X_{n,n}$), is naturally defined over $\mathbb{Q}$, we will only need a $\mathcal{K}$-rational structure.

Let $\hat{M}_H = \hat{M}(H, X_{n,n})$. Both $\hat{M}$ and $\hat{M}_H$ have $\mathcal{K}$-rational points h and h_H, respectively, and the stabilizers K_h and $K_{h,H}$ are defined over $\mathcal{K}$, as are the Harish-Chandra decompositions. For example

$$Lie(H)_{\mathcal{K}} = Lie(K_{h,H}) \oplus \mathfrak{p}_H^+ \oplus \mathfrak{p}_H^- \tag{5.1}$$

(cf. [4, 5.2] or Section 1.3 above). For example, if W has a $\mathcal{K}$-basis in terms of which the Hermitian form is given by the standard matrix $\begin{pmatrix} I_r & 0 \\ 0 & -I_s \end{pmatrix}$, then K_h is the group $K_\infty(r, s)$ defined in [6, §2.1]. For general W, we can take K_h to be the stabilizer of the diagonal Hermitian form $diag(a_1, \ldots, a_n)$ where $a_i \in \mathbb{Q}$, $a_i > 0, 1 \leq i \leq r$, $a_i < 0$, and $r + 1 \leq i \leq n$. The exact choice of Hermitian form has no bearing on rationality, though it may be relevant to integrality questions. With respect to the canonical embedding of Shimura data

$$(G(U(W) \times U(-W)), X_{r,s} \times X_{s,r}) \hookrightarrow (H, X_{n,n}),$$

we may assume $K_H \supset (K_h \times K_h) \cap H$. In other words, identifying $X_{r,s}$ with $X_{s,r}$ antiholomorphically, as in [6, (2.5.1)], we may assume that the point (h, h) maps to h_H. This antiholomorphic map is the restriction to $X_{r,s}$ of a $\mathcal{K}$-rational isomorphism

$$\phi_{r,s} : R_{\mathcal{K}/\mathbb{Q}}(X_{r,s}) \xrightarrow{\sim} R_{\mathcal{K}/\mathbb{Q}}(X_{s,r}).$$

Let K be either K_h or $K_{h,H}$, $\mathcal{G}$ correspondingly either G or H, and let (τ, V_τ) be a finite-dimensional representation of K, which can be taken rational over $\mathcal{K}$. Since the decomposition (5.1) and its analogue for G are $\mathcal{K}$-rational, the holomorphic highest weight module

$$\mathbb{D}_\tau = U(Lie(\mathcal{G})) \otimes_{U(Lie(K) \oplus \mathfrak{p}^-)} V_\tau \tag{5.2}$$

has a natural $\mathcal{K}$-rational structure; here $\mathfrak{p}^-$ is the antiholomorphic summand in the Harish-Chandra decomposition for $\mathcal{G}$. When $\mathcal{G} = H$ we only need to consider one-dimensional τ, generated by a holomorphic automorphy factor of the form $J_{\mu,\kappa}$ of (1.1.3). The main point of the comparison is that $J_{\mu,\kappa}$ is a rational function on the algebraic group H, defined over $\mathcal{K}$. The $\mathcal{K}$-rational form of $\mathbb{D}_\tau = \mathbb{D}(\mu, \kappa)$ generated by $U(\mathfrak{p}_H^+)(\mathcal{K}) \otimes J_{\mu,\kappa}$ is a $\mathcal{K}$-subspace of the space of rational functions on H.[4]

Since the rational structures on $\hat{M}$ and $\hat{M}_H$ and the corresponding Harish-Chandra decompositions are compatible, we are now in the situation considered by Garrett. Note that the representations of K_h considered in [3] arise in practice as the irreducible $U(Lie(GU(W) \times U(-W))$-summands of $\mathbb{D}(\mu, \kappa)$, as in [4, Theorem 7.4, Section 7.11]. It remains to show that the rational structure considered above is also the one used to define archimedean zeta integrals in [6]. This comes down to four points:

(1) Rationality for holomorphic Eisenstein series on H is defined in terms of rationality of the constant term, i.e., in terms of functions in the induced representation, cf. (3.1.3) and Corollary 3.2.4, as well as [6, 3.3.5.3]. By the results of [4, §5], the archimedean condition for rationality of the constant term is compatible up to $\mathcal{K}$ with rationality of $J_{\mu,\kappa}$.

(2) The $\mathcal{K}$-rational differential operators of [6, Lemma 3.3.7] are defined in terms of the $\mathcal{K}$-rational basis of $\mathfrak{p}_H^+$. This is because the *canonical trivializations* discussed in [6, §2.5] are defined in terms of the $\mathcal{K}$ rational structure of homogeneous vector bundles on $\hat{M}$ and $\hat{M}_H$. Applying this remark to the (homogeneous) normal bundle of $\hat{M} \times \hat{M}$ in $\hat{M}_H$ (cf. [4, Corollary 7.11.7]) we see that the dual basis to the $\mathcal{K}$-rational basis of $\mathfrak{p}_H^+$ chosen above can serve to define a canonical trivialization.

[4] This is not quite right as stated, because the automorphy factor has only been defined on a connected component of the real points of H. However, the explicit definition of $J_{\mu,\kappa}$ in terms of matrix entries shows that it extends to a rational function on all of H.

(3) Garrett's Theorem 2.1 calculates the local zeta integral as an E-rational multiple of π^{pq}, multiplied by the value $f(1)$, where f is a discrete-series matrix coefficient. As in [6, §3.2], the Euler product factorization of the global zeta integral is normalized in such a way as to allow us to assume that $f(1) = 1$.

(4) Finally, as already explained in (4.4)(iii), the π^{pq} in Garrett's final result is compensated by our choice of measure, so that in our normalization the zeta integral is in fact in $\mathcal{K}^\times$.

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Residues of Eisenstein Series and Related Problems

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Summary. In this paper, we discuss an approach which determines the residues of certain families of Eisenstein series in terms of periods on the cuspidal data. By contrast, the traditional approach is to determine the residues of Eisenstein series in terms of certain L-functions attached to the cuspidal data. The relation between these two methods is the general conjecture which relates periods on the cuspidal data to the existence of poles or the special values of L-functions attached to the cuspidal data.

1 Introduction

This is an extended version of my lectures at the Workshop on Eisenstein Series at The American Institute of Mathematics in August, 2006. I would like to thank the organizers Wee Teck Gan, Steve Kudla and Yuri Tschinkel for the invitation and thank the AIM for hospitality. The work is also supported in part by NSF DMS-0400414.

One of the main problems in the modern theory of automorphic forms is to understand the discrete spectrum of the space of square integrable automorphic functions on $G(\mathbb{A})$, where G is a reductive algebraic group defined over a number field k and $\mathbb{A}$ is the ring of adeles of k. It is known from the work of Gelfand and Piatetski-Shapiro and of Langlands that the discrete spectrum of the space of square integrable automorphic functions on $G(\mathbb{A})$, which is denoted by $L^2_d(G,\xi)$ (with central character ξ), can be expressed as

$$L^2_d(G,\xi) = \oplus_{\pi\in\widehat{G(\mathbb{A})}} m_d(\pi)\cdot\pi = L^2_c(G,\xi)\oplus L^2_r(G,\xi), \tag{1.1}$$

where $\widehat{G(\mathbb{A})}$ is the set of equivalence classes of irreducible admissible representations of $G(\mathbb{A})$, and $m_d(\pi)$ denotes the multiplicity of the irreducible admissible representation π of $G(\mathbb{A})$ occurring in the discrete spectrum, which is finite for each π. Also, $L^2_c(G,\xi)$ denotes the subspace of cuspidal automorphic functions, and the orthogonal complement of the cuspidal spectrum is denoted by $L^2_r(G,\xi)$, which is generated by the residues of Eisenstein series following the Langlands theory of Eisenstein series ([53]). Since the residues of Eisenstein series are constructed using the cuspidal automorphic forms on the Levi subgroups of G via a relatively simple process called parabolic induction, the structure of the residual spectrum depends more or less on the cuspidal spectrum of the smaller groups. Hence, the understanding of cuspidal automorphic forms becomes the main issue here.

On the other hand, cuspidal automorphic forms have played an important role in many related subjects, and it is hard to draw the boundaries of the theory of cuspidal automorphic forms. Within the theory of automorphic forms and representations, one may approach the problems from the following points of view:

- Whenever irreducible cuspidal automorphic representations are obtained from reasonable data, one could try to understand the cuspidal automorphic representations through these input data. The theory of theta liftings plays a role in this respect, but seems to be only effective for degenerate cuspidal automorphic representations, i.e., the CAP automorphic representations (see, e.g., [56], [59], [65], [7], [6], [36], [37], and [9]). One may think of this as understanding cuspidal automorphic representations in terms of the internal structure.
- Whenever cuspidal automorphic representations are attached to some nice invariants, it is natural to understand the cuspidal automorphic representations via these invariants. The theory of automorphic L-functions has been developed in this direction, which captures more information about the generic cuspidal automorphic representations (see [51], [33], [60], [50], and [3], for instance). One may think of this as understanding the cuspidal automorphic representations via classification by invariants.
- One may use cuspidal automorphic representations to construct relatively concrete automorphic representations of ‘bigger’ groups. Special properties of the constructed automorphic representations will provide some information about cuspidal automorphic representations we started with. The investigation of the residual spectrum should play a role in this aspect (see, e.g., [29], [30], [31], [10], [11], [12], [13], and [8]). One may think of this as understanding of cuspidal automorphic representations in terms of external structures.

In this note will focus the discussion on some recent work which can be reformulated in the third way (in terms of external structures) of understanding of cuspidal automorphic representations via the residual spectrum of the ‘bigger’ groups.

Finally, I would like to thank the referee for helpful comments.

2 Residues of Eisenstein series

Let $P = MN$ be a standard parabolic k-subgroup of G and let σ be an irreducible cuspidal automorphic representation of $M(\mathbb{A})$. The general formulation and the basic analytic properties of Eisenstein series attached to the cuspidal datum (P, σ) can be found in [53]. The main problem of interest is to determine for which cuspidal datum (P, σ) the Eisenstein series contributes nonzero residues to the discrete spectrum. This has been done for GL_n by Moeglin–Waldspurger ([52]), which was a conjecture of Jacquet [24]. For other groups, it has only been studied for k-rank two groups ([40], [41], [68], and [49]). Some discussions for higher rank groups can be found in [46], [44], [54].

2.1 CAP representations

From the point of view of Arthur's conjecture ([2]) on the basic structure of the discrete spectrum, the residual spectrum is expected to play a fundamental role in our understanding of the whole discrete spectrum. An alternative approach along the lines of Arthur's conjecture is the classification and characterization of CAP automorphic representations of $G(\mathbb{A})$. This has been recently reformulated in [36] and [37]. Hence it is worthwhile to discuss in some detail the CAP representations before concentrating on the residual spectrum.

An irreducible cuspidal automorphic representation is called CAP if it is nearly equivalent to an irreducible constituent of an induced representation from an irreducible cuspidal automorphic representation on a proper parabolic subgroup. CAP automorphic representations are cuspidal ones whose local components are expected to be non-tempered at almost all local places and hence should be counter-examples to the generalized Ramanujan conjecture. It is known that CAP automorphic representations exist in general, although Jacquet and Shalika showed that when $G = \mathrm{GL}(n)$, no such cuspidal automorphic representations exist ([23]). The existence of CAP representations makes the theory for general reductive groups more delicate than the theory for general linear groups. The CAP conjecture in [36] is stated as follows.

Conjecture 2.1 (CAP Conjecture). *Let k be a number field and $\mathbb{A}$ be the ring of adeles of k. Let G be a reductive algebraic group, which is k-quasisplit. For any given irreducible cuspidal automorphic representation π of $G(\mathbb{A})$, there exists a standard parabolic subgroup $P = MN$ of G (P may equal G) and an irreducible generic cuspidal automorphic representation σ of $M(\mathbb{A})$ such that π is nearly equivalent to an irreducible constituent of the (normalized or unitarily) induced representation*

$$I(\sigma) = \mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})}(\sigma).$$

Remark 2.2. We refer to [36] for further discussion of the CAP conjecture and give a few remarks below.

1. A version of this conjecture for $G = \mathrm{Sp}(4)$ was stated in the work of I. Piatetski-Shapiro and D. Soudry ([65] and [57]). The point here is that the CAP conjecture above requires the cuspidal datum to be generic, which is clear for the case of $\mathrm{Sp}_4(\mathbb{A})$.
2. If the parabolic subgroup P in the conjecture is proper, the representation π is a CAP automorphic representation or simply a CAP, with respect to the cuspidal data $(P; \sigma)$. If the parabolic subgroup P equals G, the representation π is nearly equivalent to an irreducible generic (having non-zero Whittaker–Fourier coefficients) cuspidal automorphic representation of $G(\mathbb{A})$.
3. The main interest for us to introduce the CAP conjecture is its relation to Arthur's conjecture on the structure of the discrete spectrum of automorphic forms. We refer to [36] for more discussion of this point.
4. One of the good features of the CAP conjecture is that the irreducible generic cuspidal datum (P, σ) attached to the irreducible cuspidal automorphic representation π of $G(\mathbb{A})$ is expected to be unique up to association class. Hence it is an invariant attached to π for classification up to nearly equivalence. See Theorem 3.2, [36] for the case of $G = \mathrm{SO}_{2n+1}$.
5. The CAP conjecture may be viewed as a global analog and generalization of the Shahidi conjecture, which asserts that every tempered local L-packet has a generic member. See our previous work on p-adic SO_{2n+1} ([38] and [35]) for the local generalization of Shahidi's conjecture. Over $\mathbb{R}$, Shahidi's conjecture was proved by D. Vogan ([67]).
6. The more general version of the CAP conjecture can be stated to take care of irreducible cuspidal automorphic representations of $G'(\mathbb{A})$ where G' is an inner form of the k-quasisplit G. We will not offer any further discussion here for this generalization of the CAP conjecture, but mention that some evidence in this aspect may be found in [57], [69], and [58].

2.2 Residues with maximal cuspidal data

In order to use the properties of the residues of Eisenstein series to study the structure of the cuspidal data (i.e., the third way to understand cuspidal automorphic representations), it is enough to consider the cases where the standard parabolic subgroup $P = MN$ is maximal in G. For simplicity of discussion, we assume that G is k-split (or may be group scheme over $\mathbb{Z}$). Let $K = \prod_v K_v$ be the standard maximal compact subgroup of $G(\mathbb{A})$ such that $G(\mathbb{A})$ has the Iwasawa decomposition

$$G(\mathbb{A}) = P(\mathbb{A})K.$$

In particular, for each finite local place v, $K_v = G(\mathcal{O}_v)$, where $\mathcal{O}_v$ is the ring of integers in the local field k_v. Then the Langlands decomposition of $G(\mathbb{A})$ is

$$G(\mathbb{A}) = N(\mathbb{A})M^1 A_M^+ K.$$

Let A_M be the (split) center of M. The unique reduced root in $\Phi^+(P, A_M)$ can be identified with simple root α. As normalized in [61], we denote

$$\tilde{\alpha} := \langle \rho_P, \alpha \rangle^{-1} \rho_P, \tag{2.1}$$

where ρ_P is half of the sum of all positive root in N and $\langle \cdot, \cdot \rangle$ is the usual Killing–Cartan form for the root system of G. We let

$$\mathfrak{a}_M = \mathrm{Hom}_{\mathbb{R}}(X(M), \mathbb{R}), \quad \mathfrak{a}_M^* = X(M) \otimes \mathbb{R}, \tag{2.2}$$

where $X(M)$ denotes the group of all rational characters of M. Since P is maximal, $\mathfrak{a}_M^*$ is of one dimension. We identify $\mathbb{C}$ with $\mathfrak{a}_{M,\mathbb{C}}^*$ via $s \mapsto s\tilde{\alpha}_M$.

Let $H_P : M \mapsto \mathfrak{a}_M$ be the map defined as follows, for any $\chi \in \mathfrak{a}_M^*$,

$$H_P(m)(\chi) = \prod_v |\chi(m_v)|_v \tag{2.3}$$

for $m \in M(\mathbb{A})$. It follows that H_P is trivial on M^1. This map H_P can be extended as a function over $G(\mathbb{A})$ via the Iwasawa decomposition or the Langlands decomposition above. Let σ be an irreducible unitary cuspidal automorphic representation of $M(\mathbb{A})$. Then (P, σ) is a cuspidal datum of $G(\mathbb{A})$. Let $\phi(g)$ be a complex-valued smooth function on $G(\mathbb{A})$ which is left $N(\mathbb{A})M(k)$-invariant and right K-finite. Writing by the Langlands decomposition

$$g = nm^1 a\kappa \in N(\mathbb{A})M^1 A_M^+ K,$$

we assume that

$$\phi(g) = \phi(m^1\kappa). \tag{2.4}$$

If we fix a $\kappa \in K$, the map

$$m^1 \mapsto \phi(m^1\kappa)$$

defines a $(K \cap M^1)$-finite vector in the space of cuspidal representation σ of $M(\mathbb{A})$. We set

$$\Phi(g; \phi, s) := H_P(g)(s + \rho_P)\phi(g).$$

Attached to such a function $\Phi(g; \phi, s)$, we define an Eisenstein series

$$E(g; \phi, s) := \sum_{\gamma \in P(k) \backslash G(k)} \Phi(\gamma g; \phi, s). \tag{2.5}$$

From the general theory of Eisenstein series [53], this Eisenstein series converges absolutely for the real part $\Re(s)$ large, and has a meromorphic continuation to the whole s-plane with finitely many possible simple poles for $\Re(s) > 0$. By the Langlands theory of the constant terms of Eisenstein series, the existence of the poles on the positive half plane of this Eisenstein series should be detected in general by means of that of the constant terms of the Eisenstein series.

By the generalized Bruhat decomposition and the cuspidality of σ, the only nonzero constant term of the Eisenstein series $E(g;\phi,s)$ is the one along the maximal parabolic subgroup P, i.e.,

$$E_P(g;\phi,s) := \int_{N(k)\backslash N(\mathbb{A})} E(ug;\phi,s)du \tag{2.6}$$

where du is the Haar measure on $N(k)\backslash N(\mathbb{A})$, which is normalized so that the total volume equals one. By assuming the real part of s large, we have

$$E_P(g;\phi,s) = \Phi(g;\phi,s) + \Phi(g;M(s,w_0)(\phi),-s) \tag{2.7}$$

where $M(s,w_0)$ is the standard (global) intertwining operator attached to the maximal parabolic subgroup P and the Weyl element w_0, which has the property that $w_0Mw_0^{-1} = M$ and $w_0Nw_0^{-1} = N^-$ (the opposite of N). This equation holds for all $s \in \mathbb{C}$ by meromorphic continuation. The intertwining operator $M(s,w_0)$ can be expressed as

$$M(s,w_0) = \otimes_v A(s,\sigma_v,w_0) \tag{2.8}$$

by following the notations in [62], where $A(s,\sigma_v,w_0)$ is the local intertwining operator

$$\mathrm{Ind}_{P(k_v)}^{G(k_v)}(\sigma_v \otimes \exp(\langle s, H_P(\cdot)\rangle)) \to \mathrm{Ind}_{P(k_v)}^{G(k_v)}(w_{2n}(\sigma_v) \otimes \exp(\langle -s, H_P(\cdot)\rangle)).$$

Then $A(s,\sigma_v,w_0)$ is normalized as follows ([62]):

$$A(s,\sigma_v,w_{2n}) = \prod_{i=1}^{m} \frac{L(is,\sigma_v,r_i)}{L(1+is,\sigma_v,r_i)\epsilon(s,\sigma_v,r_i,\psi_v)} \cdot N(s,\sigma_v,w_0). \tag{2.9}$$

One may find a clear expository discussion in [4]. We would like to mention two important local conjectures:

1. *Shahidi Conjecture ([62]):* If σ_v is tempered and generic, then the local L-functions $L(s,\sigma_v,r_i)$ occurring in (2.9) are holomorphic for $\Re(s) > 0$.
2. *Assumption (A) ([42]):* If σ is also generic (i.e., has a nonzero Whittaker–Fourier coefficient), then the normalized local intertwining operator $N(s,\sigma_v,w_0)$ is holomorphic and nonzero for $\Re(s) \geq \frac{1}{2}$ for all local places v.

Significant progress towards a proof of the two conjectures has been accomplished recently in a series of papers by H. Kim ([42], [43], [45], and [39]).

Assuming these two conjectures, it is clear that for irreducible unitary generic cuspidal automorphic representations σ of $M(\mathbb{A})$, the poles of the Eisenstein series $E(g;\phi,s)$ (which is on the real axis under the normalization of σ) are determined by the poles of the product

$$\prod_{i=1}^{m} \frac{L(is,\sigma,r_i)}{L(1+is,\sigma,r_i)}.$$

By the argument of Kim and Shahidi ([48] and [47]), it is expected that for $i \geq 3$, the L-functions $L(s, \sigma, r_i)$ are holomorphic for $\Re(s) > \frac{1}{2}$ and do not vanish for $\Re(s) \geq 1$. It is also expected that for $i = 1, 2$, the L-functions $L(s, \sigma, r_i)$ have at most a simple pole at $s = 1$. These statements have been proved in all cases for the automorphic L-functions in the Langlands–Shahidi list, with a few exceptions. We refer to [48] and [47] for more details.

It follows that for an irreducible generic unitary maximal cuspidal data (P, σ), the Eisenstein series $E(g; \phi, s)$ is expected to have at most a simple pole at $s = \frac{1}{2}$ or $s = 1$. We denote the residues of the Eisenstein series by

$$\mathcal{E}_{\frac{1}{2}}(g; \phi) = \operatorname{Res}_{s=\frac{1}{2}} E(g; \phi, s) \tag{2.10}$$

$$\mathcal{E}_1(g; \phi) = \operatorname{Res}_{s=1} E(g; \phi, s). \tag{2.11}$$

Concerning these residues of Eisenstein series, we would like to address the following problems, which have been investigated for many interesting and important cases.

1. For an irreducible generic unitary maximal cuspidal data (P, σ), the possible simple poles of the Eisenstein series $E(g; \phi, s)$ can only be at either $s = \frac{1}{2}$ or $s = 1$, but not at both.
2. Determine the structure of the irreducible generic unitary maximal cuspidal data (P, σ) according to the location of the possible simple pole of the Eisenstein series $E(g; \phi, s)$.

Rather than giving a conceptual approach to these problems, we would like to explain our ideas for specific cases which seem to provide satisfactory answers.

3 Local theory: generic reducibility

For a finite local place v of k, let σ_v be an irreducible generic unitary supercuspidal representation of $M(k_v)$. Consider the normalized induced representation

$$\mathrm{I}(s, \sigma_v) = \operatorname{Ind}_{P(k_v)}^{G(k_v)}(\sigma_v \otimes \exp(< s, H_P(\cdot) >)). \tag{3.1}$$

It is proved in [62] that the reducibility point with $s \geq 0$ for the induced representation $\mathrm{I}(s, \sigma_v)$ can occur at one and only one of the s in $\{0, \frac{1}{2}, 1\}$. This unique reducibility of $\mathrm{I}(s, \sigma_v)$ is found to be important in [62] for the determination of the complimentary series representations attached to the supercuspidal datum (P, σ_v). The essence of the series of papers [63], [64], [15] and [16] is to understand the reducibility point of $\mathrm{I}(s, \sigma_v)$ in terms of certain orbital integrals attached to the supercuspidal datum (P, σ_v), which has been successful with respect to $s = 1$.

It is also natural to try to understand the reducibility of (P, σ_v) in terms of poles related to local γ-factors, or in terms of the explicit local Langlands transfer from G to the general linear group via the standard representation of the Langlands dual group ${}^LG$. To explain these ideas, we may take, for instance, $G = \mathrm{SO}_{2(r+l)+1}$.

Let $P_{r,n} = M_{r,l}N_{r,l}$ be the maximal parabolic subgroup of $\mathrm{SO}_{2(r+l)+1}$ with the Levi subgroup $M_{r,l}$ isomorphic to $\mathrm{GL}_r \times \mathrm{SO}_{2l+1}$, so that we write an element of $M_{r,l}$ as $m = m(a,h)$, with $a \in \mathrm{GL}_r$ and $h \in \mathrm{SO}_{2l+1}$. We write elements of $N_{r,l}$ as

$$n = n(x,z) = \begin{pmatrix} I_r & x & z \\ & I_{2l+1} & x^* \\ & & I_r \end{pmatrix} \in \mathrm{SO}_{2(r+l)+1}\,. \tag{3.2}$$

Let τ_v be an irreducible unitary supercuspidal representation of $\mathrm{GL}_r(k_v)$ and π_v be an irreducible generic unitary supercuspidal representation of $\mathrm{SO}_{2l+1}(k_v)$. Then we have a generic supercuspidal datum $(P_{r,l}, \tau_v \otimes \pi_v)$ of $\mathrm{SO}_{2(r+l)+1}(k_v)$. The induced representation we are going to consider is

$$\mathrm{I}(s, \tau_v \otimes \pi_v) = \mathrm{Ind}_{P_{r,l}(k_v)}^{\mathrm{SO}_{2(r+l)+1}(k_v)}(\tau_v \cdot |\det(a)|^s \otimes \pi_v). \tag{3.3}$$

Recall from [38] and [35] the explicit local Langlands functorial transfer from irreducible generic representations of $\mathrm{SO}_{2l+1}(k_v)$ to $\mathrm{GL}_{2l}(k_v)$. The image of the local Langlands transfer of the irreducible unitary generic supercuspidal representation π_v of $\mathrm{SO}_{2l+1}(k_v)$ is of the form

$$\tau(\pi_v) = \tau_{1,v} \boxplus \cdots \boxplus \tau_{m,v} \tag{3.4}$$

with the properties that

1. for each i, $\tau_{i,v}$ is an irreducible supercuspidal representation of $\mathrm{GL}_{2l_i}(k_v)$, with $l = \sum_{i=1}^m l_i$ and $\tau_{i,v} \not\cong \tau_{j,v}$ if $i \neq j$;
2. for each i, the local exterior square L-factor $L(s, \tau_{i,v}, \Lambda^2)$ has a pole (which is simple) at $s = 0$, or equivalently, the local γ-factor $\gamma(s, \tau_{i,v}, \Lambda^2, \psi_v)$ has a simple pole at $s = 1$.

From this explicit local Langlands functorial transfer, we expect that the following holds:

(C1) The induced representation $\mathrm{I}(s, \tau_v \otimes \pi_v)$ reduces at $s = 1$ if and only if the supercuspidal representation τ_v is equivalent to one of the supercuspidal representations $\tau_{1,v}, \cdots, \tau_{m,v}$, or equivalently, the local Rankin–Selberg L-factor $L(s, \tau_v \times \pi_v)$ has a pole at $s = 0$. In particular, τ_v must be self-dual.

(C$\frac{1}{2}$) The induced representation $\mathrm{I}(s, \tau_v \otimes \pi_v)$ reduces at $s = \frac{1}{2}$ if and only if τ is self-dual with the property that the local symmetric square L-factor $L(s, \tau_v, \mathrm{Sym}^2)$ has a pole at $s = 0$, or equivalently the local γ-factor $\gamma(s, \tau_v, \mathrm{Sym}^2, \psi_v)$ has a pole at $s = 1$, and the pair (τ_v, π_v) shares a nonzero Gross–Prasad functional, which is defined below.

(C0) The induced representation $\mathrm{I}(s, \tau_v \otimes \pi_v)$ reduces at $s = 0$ if and only if the pair (τ_v, π_v) does not satisfy any of the conditions above.

It remains to define the Gross–Prasad functionals for the pair (τ_v, π_v). Since τ_v is an irreducible unitary supercuspidal representation of $\mathrm{GL}_r(k_v)$ with the property that the local symmetric square L-factor $L(s, \tau_v, \mathrm{Sym}^2)$ has a pole at $s = 0$, we deduce easily from the automorphic descent construction of Ginzburg–Rallis–Soudry ([66]) that

Even if r is even, then there exists an irreducible generic unitary supercuspidal representation σ_v of $\mathrm{SO}_r(k_v)$ such that τ_v is the image of σ_v under the local Langlands functorial transfer; and

Odd if r is odd, then there exists an irreducible generic unitary supercuspidal representation σ_v of $\mathrm{Sp}_{r-1}(k_v)$ such that τ_v is the image of σ_v under the local Langlands functorial transfer.

On the other hand, if the image of π_v under the local Langlands functorial transfer is given by (3.4), then it follows from [38] and [35] that there exist irreducible generic unitary supercuspidal representations $\pi_{1,v}, \cdots, \pi_{m,v}$ such that π_v is the image of $\pi_{1,v} \otimes \cdots \otimes \pi_{m,v}$ under the local Langlands functorial transfer from

$$\mathrm{SO}_{2l_1+1}(k_v) \times \cdots \times \mathrm{SO}_{2l_m+1}(k_v)$$

to

$$\mathrm{SO}_{2(l_1+\cdots+l_m)+1}(k_v) = \mathrm{SO}_{2l+1}(k_v).$$

We say that the pair (τ_v, π_v) shares a nonzero Gross–Prasad functional if the pair $(\sigma_v, \pi_{i,v})$ shares a nonzero Gross–Prasad functional for some $i \in \{1, 2, \cdots, m\}$. When r is even, the Gross–Prasad functions for the pair $(\sigma_v, \pi_{i,v})$ of $\mathrm{SO}_r \times \mathrm{SO}_{2l_i+1}$ is the usual Gelfand–Graev functional or model as defined in [17] and [18]. When r is odd, first we have to use the local theta lifting of $\pi_{i,v}$ to $\tilde{\pi}_{i,v}$ of $\widetilde{\mathrm{Sp}}_{2l_i}(k_v)$ and then we consider the local Fourier–Jacobi functional or model for the pair $(\sigma_v, \tilde{\pi}_{i,v})$ of $\mathrm{Sp}_{2l_i} \times \widetilde{\mathrm{Sp}}_{2l_i}$. The global version of the Fourier–Jacobi models and the analog of the Gross–Prasad conjecture have been extensively discussed in [11]. We omit the detailed discussion of the local version.

Of course, the reducibility of $\mathrm{I}(s, \tau_v \otimes \pi_v)$ at $s = 1$ can also be characterized in terms of the nonvanishing of a certain linear functional. By the hereditary property ([5]), one expects that the Langlands quotient of the $\mathrm{I}(s, \tau_v \otimes \pi_v)$ has a nonzero functional of the same type. The global version of this claim has been carried out in [11] and [13].

4 Models for residual representations

We will discuss in this section how the structure of the cuspidal data is expected to determine the location of the poles of the Eisenstein series attached to the cuspidal data. We give some examples to explain the ideas and the methods used in our work.

4.1 Periods of Gross–Prasad type

For G_n to be one of the k-split groups SO_{2n+1}, $\mathrm{Sp}_2 n$, and SO_{2n}, let $P_{r,l} = M_{r,l}N_{r,l}$ be the standard maximal parabolic subgroup of G_n with $n = r + l$ and $l \geq 1$. The location of the pole of the Eisenstein series $E(g; \phi_\sigma, s)$ attached to the generic cuspidal datum $(P_{r,l}, \sigma)$ has been determined in [11] and [13] in terms of the nonvanishing period of the Gross–Prasad type.

Theorem 4.1 ([11] and [13]). *In the above cases, if the generic cuspidal data $(P_{r,l}, \sigma)$ has a nonzero period of the Gross–Prasad type, then the Eisenstein series $E(g; \phi_\sigma, s)$ must have a pole at $s = \frac{1}{2}$ and has no other poles for $\Re(s) \geq 0$. Moreover the residue $\mathcal{E}_{\frac{1}{2}}(g; \phi_\sigma)$ has a nonzero period of the same type.*

We refer to [11] and [13] for details concerning the definition of the periods of the Gross–Prasad type, i.e., of either the Gelfand–Graev type in the orthogonal group case or the Fourier–Jacobi type in the symplectic group case. We note that in both [11] and [13], we have an assumption on the generic cuspidal data. More precisely, we write $\sigma = \tau \otimes \pi$ with π being an irreducible unitary generic cuspidal automorphic representation of G_l (which is the part of the Levi subgroup $M_{r,l}$). The assumption is that the image of π under the Langlands functorial transfer to the general linear group is cuspidal. This is just a technical assumption. Our ideas and methods used in [11] and [13] are expected to work in the general case without this assumption.

We also refer to [32] for a general discussion of periods of automorphic forms. The proof of an analog of Theorem 4.1 for k-quasisplit unitary groups has been discussed in [8]. Some other examples have been worked out in [29], [10], [12].

4.2 Generalized Shalika periods

We discuss the Shalika periods and related results. Let τ be an irreducible unitary cuspidal automorphic representation of $\mathrm{GL}_{2n}(\mathbb{A})$. Let $\mathcal{S}_n$ be the Shalika subgroup of GL_{2n} consisting of elements of the following type

$$s = s(x, h) = \begin{pmatrix} I_n & x \\ & I_n \end{pmatrix} \begin{pmatrix} h & \\ & h \end{pmatrix} \in \mathrm{GL}_{2n}.$$

Fix a nontrivial character ψ of $\mathbb{A}/k$ and define a character $\psi_{\mathcal{S}_n}$ by

$$\psi_{\mathcal{S}_n}(s(x, h)) := \psi(\mathrm{tr}(x)). \tag{4.1}$$

It is easy to check that the character $\psi_{\mathcal{S}_n}$ originally defined for the unipotent part of $\mathcal{S}_n$ can be extended to the whole group $\mathcal{S}_n$. For any $\phi_\tau \in \tau$, the Shalika period is defined ([28]) by

$$\mathcal{P}_{\mathcal{S}_n,\psi_{\mathcal{S}_n}}(\phi_\tau) := \int_{\mathcal{S}_n(k)\backslash\mathcal{S}_n(\mathbb{A})} \phi_\tau(s(x,h))\psi_{\mathcal{S}_n}^{-1}(x)dxdh. \tag{4.2}$$

It is proved in [28] that the Shalika period for τ is related to the simple pole of the exterior square L-function $L(s,\tau,\Lambda^2)$. More precisely we recall from [33] the following important properties.

Theorem 4.2. *Let τ be an irreducible, unitary, self-dual, cuspidal automorphic representation of $\mathrm{GL}_{2n}(\mathbb{A})$.*

(1) The exterior square L-function $L(s,\tau,\Lambda^2)$ is holomorphic for $\Re(s) > 1$, and nonvanishing for $\Re(s) \geq 1$ ([39], Proposition 8.2).
(2) $L(s,\tau,\Lambda^2)$ has at most a simple pole at $s = 1$.
(3) $L(s,\tau,\Lambda^2)$ has a simple pole at $s = 1$ if and only if τ is a Langlands functorial lifting from an irreducible generic cuspidal automorphic representation σ of $\mathrm{SO}_{2n+1}(\mathbb{A})$.
(4) Let S be a finite set of local places of k including all archimedean places. The complete exterior square L-function $L(s,\tau,\Lambda^2)$ has a simple pole at $s = 1$ if and only if the partial exterior square L-function $L^S(s,\tau,\Lambda^2)$ has a simple pole at $s = 1$.
(5) The partial exterior square L-function $L^S(s,\tau,\Lambda^2)$ has a simple pole at $s = 1$ if and only if the Shalika period $\mathcal{P}_{\mathcal{S}_n,\psi_{\mathcal{S}_n}}(\phi_\tau)$ does not vanish for some $\phi_\tau \in \tau$ (the main theorem of [28]).

This is the restatement of Theorem 2.2 in [33]. We refer to [33] for the discussion of the proofs of each property. We would like to thank Henry Kim for pointing out that the first property is completely proved in his recent preprint ([39]).

Next, we will discuss the generalization of the Shalika period to the k-split even special orthogonal group SO_{4n}, which determines the location of the pole of the Eisenstein series $E(g;\phi_\tau,s)$ attached to the cuspidal datum (P_{2n},τ), where $P_{2n} = M_{2n}N_{2n}$ is the parabolic subgroup with Levi part M_{2n} isomorphic to GL_{2n}. Assume that SO_{4n} is defined with respect to the quadratic form attached to the symmetric matrix, given inductively by

$$J_{4n}^+ = \begin{pmatrix} & & 1 \\ & J_{4n-2}^+ & \\ 1 & & \end{pmatrix}.$$

Then the element of N_{2n} can be written as

$$n = n(X) = \begin{pmatrix} I_{2n} & X \\ & I_{2n} \end{pmatrix} \in N_{2n} \in \mathrm{SO}_{4n}\,.$$

We denote by J_{2n}^- the skew-symmetric matrix, given inductively by

$$J_{2n}^- = \begin{pmatrix} & & 1 \\ & J_{2n-2}^- & \\ -1 & & \end{pmatrix}.$$

We define a character $\psi_{N_{2n}}$ of N_{2n} by

$$\psi_{N_{2n}}(n(X)) := \psi(\mathrm{tr}(XJ_{2n}^{-})).$$

It is clear that the stabilizer of $\psi_{N_{2n}}$ in M_{2n} is isomorphic to Sp_{2n}. We define

$$\mathcal{R}_{2n} := N_{2n} \rtimes \mathrm{Sp}_{2n}. \tag{4.3}$$

It is clear that the character ψ_{2n} can be extended to be a character $\psi_{\mathcal{R}_{2n}}$ of $\mathcal{R}_{2n}$ by

$$\psi_{\mathcal{R}_{2n}}(r(X,h)) := \psi_{N_{2n}}(n(X)). \tag{4.4}$$

The generalized Shalika period for any automorphic form ϕ of $\mathrm{SO}_{4n}(\mathbb{A})$ is defined by

$$\mathcal{P}_{\mathcal{R}_{2n},\psi_{\mathcal{R}_{2n}}}(\phi) := \int_{\mathcal{R}_{2n}(k)\backslash\mathcal{R}_{2n}(\mathbb{A})} \phi(r(X,h))\psi^{-1}_{\mathcal{R}_{2n}}(r(X,h))dXdh. \tag{4.5}$$

When ϕ is not cuspidal, the period may diverge, and so a certain regularization is needed for the period to be well defined (see [1], [27], [26], and [25] for various truncation methods). Details for the current case are given in [34]. We also refer to [12] for a treatment of the regularization in similar cases. We would like to thank David Ginzburg and Steve Rallis for discussions about the generalization of Shalika models to other groups. The following results are proved in [34].

Theorem 4.3 ([34]). *Let τ be an irreducible unitary cuspidal automorphic representation of $\mathrm{GL}_{2n}(\mathbb{A})$. Assume that τ has nontrivial Shalika periods $\mathcal{P}_{\mathcal{S}_n,\psi_{\mathcal{S}_n}}(\phi_\tau)$. Then the Eisenstein series $E(g;,\phi_\tau,s)$ on $\mathrm{SO}_{4n}(\mathbb{A})$ attached to the cuspidal datum (P_{2n},τ) has at most a simple pole at $s=1$. Moreover, the residue at $s=1$, which is denoted by $\mathcal{E}_1(g;\phi_\tau)$, of the Eisenstein series $E(g;,\phi_\tau,s)$ has a nontrivial generalized Shalika period $\mathcal{P}_{\mathcal{R}_{2n},\psi_{\mathcal{R}_{2n}}}(\mathcal{E}_1(g;\phi_\tau))$.*

The local p-adic version can be stated as follows. The induced representation is

$$\mathrm{I}(s,\tau_v) := \mathrm{Ind}_{P_{2n}(k_v)}^{\mathrm{SO}_{4n}(k_v)}(\tau_v \otimes \exp(< s, H_{P_{2n}}(\cdot) >)).$$

Theorem 4.4 ([34]). *With the notation above, the induced representation $\mathrm{I}(s,\tau_v)$ admits a nontrivial local generalized Shalika functional if and only if τ_v is self-dual and $s=1$. In this case, the nontrivial local generalized Shalika functional factors through the unique Langlands quotient of $\mathrm{I}(s,\tau_v)$ at $s=1$.*

The local generalized Shalika functional can be easily defined according the generalized Shalika periods defined in (4.5), and we refer to [34] for details.

4.3 The Ginzburg–Rallis periods

In [14], Ginzburg and Rallis found an integral representation for the exterior cube L-function $L(s,\tau,\Lambda^3)$ attached to the irreducible cuspidal automorphic representation τ of $\mathrm{GL}_6(\mathbb{A})$ and the exterior cube complex representation Λ^3 of $\mathrm{GL}_6(\mathbb{C})$, and then by using a regularized Siegel–Weil formula, they proved that the nonvanishing of the central value of the exterior cube L-function, $L(\frac{1}{2},\tau,\Lambda^3)$ is related to the following period, which we call the Ginzburg–Rallis period.

Let $P_{2,2,2}=P=MN$ be the standard parabolic subgroup of GL_6 with the Levi part M isomorphic to $\mathrm{GL}_2\times\mathrm{GL}_2\times\mathrm{GL}_2$. The element of N is of the form

$$n=n(x,y,z)=\begin{pmatrix} I_2 & x & z\\ & I_2 & y\\ & & I_2\end{pmatrix}\in\mathrm{GL}_6.$$

Define a character of N by

$$\psi_N(n(x,y,z))=\psi(\mathrm{tr}(x+y)). \tag{4.6}$$

The stabilizer of $\psi_N$in M is the diagonal embedding of GL_2. Consider the subgroup $\mathcal{H}$ of GL_6 consisting of elements of following type:

$$h(n(x,y,z),g)=\begin{pmatrix} I_2 & x & z\\ & I_2 & y\\ & & I_2\end{pmatrix}\cdot\begin{pmatrix} g & & \\ & g & \\ & & g\end{pmatrix}\in\mathrm{GL}_6, \tag{4.7}$$

with $g\in\mathrm{GL}_2$. Then the character ψ_N can be extended to a character $\psi_{\mathcal{H}}$ of $\mathcal{H}$. The Ginzburg–Rallis period for automorphic forms ϕ on $\mathrm{GL}_6(\mathbb{A})$ is defined to be

$$\mathcal{P}_{\mathcal{H},\psi_{\mathcal{H}}}(\phi):=\int_{\mathcal{H}(k)\backslash\mathcal{H}(\mathbb{A})}\phi(h)\psi_{\mathcal{H}}^{-1}(h)dh. \tag{4.8}$$

In [14], they also introduce a nonsplit version of the above period. Take a quaternion division algebra $\mathcal{D}$ over k. In $\mathrm{GL}_3(\mathcal{D})$, consider the following elements

$$h_{\mathcal{D}}(n(x,y,z),g):=\begin{pmatrix} 1 & x & z\\ & 1 & y\\ & & 1\end{pmatrix}\cdot\begin{pmatrix} g & & \\ & g & \\ & & g\end{pmatrix}\in\mathrm{GL}_3(\mathcal{D}), \tag{4.9}$$

with $g\in\mathcal{D}^\times$, elements form a k-subgroup of $\mathrm{GL}_3(\mathcal{D})$, denoted by $\mathcal{H}_{\mathcal{D}}$. Let $\psi_{\mathcal{H}_{\mathcal{D}}}$ be the character of $\mathcal{H}_{\mathcal{D}}$ defined by

$$\psi_{\mathcal{H}_{\mathcal{D}}}(h_{\mathcal{D}}(n(x,y,z),g)):=\psi(\mathrm{tr}_{\mathcal{D}/k}(x+y)), \tag{4.10}$$

where $\mathrm{tr}_{\mathcal{D}/k}$ is the reduced norm of $\mathcal{D}$. Then the nonsplit version of the Ginzburg–Rallis period for an automorphic form $\phi^{\mathcal{D}}$ on $\mathrm{GL}_3(\mathcal{D})$ is defined to be

$$\mathcal{P}_{\mathcal{H}_{\mathcal{D}},\psi_{\mathcal{H}_{\mathcal{D}}}}(\phi^{\mathcal{D}}):=\int_{\mathcal{H}_{\mathcal{D}}(k)\backslash\mathcal{H}_{\mathcal{D}}(\mathbb{A})}\phi^{\mathcal{D}}(h_{\mathcal{D}})\psi_{\mathcal{H}_{\mathcal{D}}}^{-1}(h_{\mathcal{D}})dh_{\mathcal{D}}. \tag{4.11}$$

Conjecture 4.5 ([14]). *For an irreducible cuspidal representation τ of $\mathrm{GL}_6(\mathbb{A})$ and for a character χ of $\mathbb{A}^\times/k^\times$ with the property that $\omega_\tau \cdot \chi^2 = 1$, the central value of the exterior cube L-function $L(\frac{1}{2}, \tau, \Lambda^3)$ does not vanish if and only if there exists a unique quaternion algebra $\mathcal{D}$ over k and a Jacquet–Langlands correspondence $\tau^{\mathcal{D}}$ of τ from $\mathrm{GL}_6(\mathbb{A})$ to $\mathrm{GL}_3(\mathcal{D})(\mathbb{A})$ such that the period $\mathcal{P}_{\mathcal{H}_{\mathcal{D}},\psi_{\mathcal{H}_{\mathcal{D}}}}(\phi^{\mathcal{D}})$ does not vanish for some $\phi^{\mathcal{D}} \in \tau^{\mathcal{D}}$, and the period $\mathcal{P}_{\mathcal{H}_{\mathcal{D}'},\psi_{\mathcal{H}_{\mathcal{D}'}}}(\phi^{\mathcal{D}'})$ vanishes identically for all quaternion algebras $\mathcal{D}'$ not isomorphic to $\mathcal{D}$ over k.*

The conjecture of Ginzhurg and Rallis is a beautiful analog of the Jacquet conjecture for the trilinear periods and the triple product L-functions of GL_2, which is now a theorem of Harris and Kudla ([19]). In [14], the conjecture was made for the partial L-function, instead of the complete L-function. By the local Langlands conjecture for GL_n proved by Harris–Taylor ([21]) and Henniart ([22]), and by the work of Langlands at the archimedean local places, it is possible to define the local L-factors at the unramified places. If the local component τ_v at a ramified local place is non-tempered, one may worry about the possibility of occurrence of a pole at $s = \frac{1}{2}$. Assuming the generalized Ramanujan conjecture for cuspidal automorphic representations of GL_n, it makes sense to use the complete L-function in the conjecture.

It is also known that the twisted exterior cube L-functions occur in the Langlands–Shahidi list ([61]), in particular, it is the case when G is the simply connected k-split group of type E_6 and the Levi part is of type A_5. In [48], some refined properties for the twisted exterior cube L-functions have been established. It is natural to determine the poles of the Eisenstein series in this case in terms of the Ginzburg–Rallis period on the cuspidal datum. This is a work in progress by the author and David Ginzburg.

It is easy to see that the exterior cube L-function for GL_6 is of symplectic type. Hence the central value of the L-function should be critical in the sense of motives. This motivates the investigation of the local theory for the Ginzburg–Rallis periods. In her PhD thesis 2006, C.-F. Nien proved the local uniqueness of the Ginzburg–Rallis functional for irreducible admissible representations of $\mathrm{GL}_3(\mathcal{D}_v)$ over p-adic local fields k_v, where $\mathcal{D}_v$ is a quaternion algebra over k_v which can be either division or k_v-split ([55]). It is a work in progress to prove the following reasonable and beautiful conjecture.

Conjecture 4.6. *For any irreducible admissible representation τ_v of $\mathrm{GL}_6(k_v)$, define $\tau_v^{\mathcal{D}_v}$ to be the Jacquet–Langlands correspondence of τ_v to $\mathrm{GL}_3(\mathcal{D}_v)$ if it exists for the unique quaternion division algebra $\mathcal{D}_v$ over k_v, and to be zero, otherwise. Then the following holds*

$$\dim \mathrm{Hom}_{\mathcal{H}(k_v)}(\tau_v, \psi_{\mathcal{H}_v}) + \dim \mathrm{Hom}_{\mathcal{H}_{\mathcal{D}_v}(k_v)}(\tau_v^{\mathcal{D}_v}, \psi_{\mathcal{H}_{\mathcal{D}_v}}) = 1.$$

It is also expected that the ϵ-dichotomy from [20] holds for the local Ginzburg–Rallis functionals.

5 Final Remarks

To summarize, there are two ways to determine the location of poles of Eisenstein series in terms of basic properties of given cuspidal data. One is to use the information coming from automorphic L-functions occurring in constant terms of the Eisenstein series. This approach is based on the Langlands theory of Eisenstein series and their constant terms. The problem reduces to a local problem on the normalization of local intertwining operators and the local reducibility problem of the standard modules. The other approach is to use periods of automorphic forms. This method is motivated by the classical representations and invariants of algebraic groups, and has the same flavor as the Rankin–Selberg method for automorphic L-functions. We would like to mention that the connection between these two approaches is the so called relation between periods of automorphic forms and special values or poles of automorphic L-functions. Some interesting and important cases for classical groups have been formulated in the framework of the Gross–Prasad conjecture. More interesting conjectures of this type are expected to be discovered in future work.

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Some Extensions of the Siegel–Weil Formula

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Summary. We survey recent extensions of the classical Siegel–Weil formula linking special values of certain Eisenstein series with integrals of theta functions to relations between residues of Eisenstein series and regularized integrals of theta functions.

Introduction

In this article I will survey some relatively recent joint work with S. Rallis, in which we extend the classical formula of Siegel and Weil. In the classical case, this formula identifies a special value of a certain Eisenstein series as an integral of a theta function. Our extension identifies the residues of the (normalized) Eisenstein series on $\mathrm{Sp}(n)$ as 'regularized' integrals of theta functions. Moreover, we obtain an analogous result for the value of the Eisenstein series at its center of symmetry (when n is odd). Both of these identities have applications to special values and to poles of Langlands L-functions. Most of our results were announced by Rallis in his lecture [36] at the ICM in Kyoto in August of 1990, and detailed proofs will appear shortly [24]. Thus the present article will be mostly expository. However, some of the results of Section 3.4 about 'second term identities' have not appeared elsewhere. In the last section, I have explained how some of our results may be translated into a more classical language.

This article[1] is an expansion of the two hour lecture I gave at RIMS, Kyoto in January of 1992 as part of the conference Studies on Automorphic Forms and Associated L-Functions, organized by K. Takase. I would like to thank Takase for his fine job in organizing the conference and for his essential assistance in arranging my visit to Japan. I would like to thank T. Oda,

[1] This article appeared in the proceedings of that conference, but was not widely circulated. I have made no attempt to seriously revise or update it, but hope that it may still provide a useful introduction to the regularized Siegel–Weil formula. I would like to thank the referee for pointing out a number of typos and suggesting several improvements in the text.

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1

1.1 Background

We will work adelically and recall some of the basic machinery of theta functions, Eisenstein series, etc.

Let F be a totally real number field and fix a non-trivial additive character ψ of $F_{\mathbb{A}}/F$. Let V, $(\ ,\ )$ be a non-degenerate inner product space over F with $\dim_F V = m$. For convenience, we will assume that m is even. Associated to V is a quadratic character χ_V of $F_{\mathbb{A}}^\times/F^\times$,

$$\chi_V(x) = (x, (-1)^{\frac{m}{2}} \det(V))_F \tag{1.1}$$

where $(\ ,\ )_F$ is the Hilbert symbol of F and $\det(V) \in F^\times/F^{\times,2}$ is the determinant of the matrix $((x_i, x_j))$ for any basis $x_1, \ldots, x_m$ of V. Let $H = \mathrm{O}(V)$ be the orthogonal group of V. Also let: $W = F^{2n}$ (row vectors); $\langle\ ,\ \rangle$ be the standard symplectic vector space over F; and $G = \mathrm{Sp}(W) = \mathrm{Sp}(n)$ be the corresponding symplectic group. The Siegel parabolic $P = MN$ of G is the subgroup which stabilizes the maximal isotropic subspace $\{\ (0, y) \mid y \in F^n\ \}$. It has a Levi factor

$$M = \left\{\ m(a) = \begin{pmatrix} a & 0 \\ 0 & {}^t a^{-1} \end{pmatrix} \mid a \in \mathrm{GL}(n)\ \right\}, \tag{1.2}$$

and a unipotent radical

$$N = \left\{\ n(b) = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \mid b = {}^t b \in \mathrm{Sym}(n)\ \right\}. \tag{1.3}$$

For each non-archimedean place v of F, let $K_v = \mathrm{Sp}(n, \mathcal{O}_v)$ where $\mathcal{O}_v$ is the ring of integral elements in the completion F_v. If v is an archimedean place of F let

$$K_v = \left\{\ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \mid a + ib \in \mathrm{U}(n)\ \right\}. \tag{1.4}$$

Then let $K = \prod_v K_v$ be the corresponding maximal compact subgroup of $G(\mathbb{A})$.

Let $\mathbb{W} = V \otimes_F W \simeq V^{2n}$ with symplectic form $\langle\langle\ ,\ \rangle\rangle = (\ ,\ ) \otimes \langle\ ,\ \rangle$, so that (G, H) is a dual reductive pair in $\mathrm{Sp}(\mathbb{W})$.

Let $\mathrm{Mp}(\mathbb{W}) \longrightarrow \mathrm{Sp}(\mathbb{W})(\mathbb{A})$ be the metaplectic cover of $\mathrm{Sp}(\mathbb{W})(\mathbb{A})$ and let $\omega = \omega_\psi$ be the Weil representation [40] of $\mathrm{Mp}(\mathbb{W})$ associated to ψ. Since we are assuming that $m = \dim_F(V)$ is even, there is a splitting

$$\begin{array}{ccc} & & \mathrm{Mp}(\mathbb{W}) \\ & \nearrow & \downarrow \\ G(\mathbb{A}) \times H(\mathbb{A}) & \longrightarrow & \mathrm{Sp}(\mathbb{W})(\mathbb{A}) \end{array} \tag{1.5}$$

and ω yields a representation of $G(\mathbb{A}) \times H(\mathbb{A})$. This can be realized on the space $S(V(\mathbb{A})^n)$ of Schwartz–Bruhat functions on $V(\mathbb{A})^n$ with

$$\big(\omega(h)\varphi\big)(x) = \varphi(h^{-1}x) \qquad h \in H(\mathbb{A}) \tag{1.6}$$

$$\big(\omega(m(a))\varphi\big)(x) = \chi_V(x)|\det(a)|_{\mathbb{A}}^{\frac{m}{2}}\,\varphi(xa) \qquad a \in \mathrm{GL}_n(\mathbb{A}) \tag{1.7}$$

$$\big(\omega(n(b))\varphi\big)(x) = \psi(\mathrm{tr}(bQ(x)))\varphi(x) \qquad b \in \mathrm{Sym}_n(\mathbb{A}). \tag{1.8}$$

Here $Q(x) = \frac{1}{2}(x,x)$ is the value on $x \in V(\mathbb{A})^n$ of the quadratic form Q associated to (,).

1.2 Averages of theta functions

From this machinery we can construct the usual theta function. For $g \in G(\mathbb{A})$, $h \in H(\mathbb{A})$ and $\varphi \in V(\mathbb{A})^n$, let

$$\theta(g,h;\varphi) = \sum_{x \in V(F)^n} \big(\omega(g)\varphi\big)(h^{-1}x). \tag{1.9}$$

This function is left invariant under $G(F)$ (Poisson summation) and $H(F)$ and is slowly increasing on $G(F)\backslash G(\mathbb{A})$ and $H(F)\backslash H(\mathbb{A})$. The average value with respect to H is given by the integral

$$I(g;\varphi) = \int_{H(F)\backslash H(\mathbb{A})} \theta(g,h;\varphi)\,dh \tag{1.10}$$

where dh is the invariant measure on $H(F)\backslash H(\mathbb{A})$ normalized to have total volume 1 (we exclude the case of a split binary space V). Weil's convergence criterion [41] asserts that $I(g;\varphi)$ is absolutely convergent for all φ provided

$$\begin{cases} (1) & V \text{ is anisotropic or} \\ (2) & m - r > n + 1 \end{cases} \tag{1.11}$$

where r is the Witt index of V, i.e., the dimension of a maximal isotropic F-subspace of V. Note that when V is split, $m = 2r$ and condition (1.11)(2) becomes $m > 2n+2$. When (1.11)(1) or (1.11)(2) is satisfied, then $I(g;\varphi)$ is an automorphic form on $G(\mathbb{A})$, provided we assume that the function φ is K-finite.

1.3 Siegel's Eisenstein series

The Eisenstein series involved in the Siegel–Weil formula are defined as follows. The group $G(\mathbb{A})$ has an Iwasawa decomposition

$$G(\mathbb{A}) = P(\mathbb{A})K.$$

Write any $g \in G(\mathbb{A})$ as $g = nm(a)k$, and set

$$|a(g)| = |\det(a)|_{\mathbb{A}}. \tag{1.12}$$

This is a well-defined function on $G(\mathbb{A})$ which is left $N(\mathbb{A})M(F)$-invariant and right K-invariant. For $s \in \mathbb{C}$ and $\varphi \in S(V(\mathbb{A})^n)$, set

$$\Phi(g,s) = \omega(g)\varphi(0) \cdot |a(g)|^{s-s_0} \tag{1.13}$$

where

$$s_0 = \frac{m}{2} - \rho_n \tag{1.14}$$

with $\rho_n = \frac{n+1}{2}$. Note that since

$$\omega(nm(a)g)\varphi(0) = \chi_V(\det(a))|\det(a)|^{\frac{m}{2}}\omega(g)\varphi(0), \tag{1.15}$$

we have

$$\Phi(nm(a)g,s) = \chi_V(\det(a))|\det(a)|^{s+\rho_n}\Phi(g,s). \tag{1.16}$$

The space of *all* smooth K-finite functions $\Phi(s)$ on $G(\mathbb{A})$, with such a transformation on the left, form an induced representation

$$I_n(s,\chi_V) = \mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})}(\chi_V \cdot |\ |^s), \qquad \text{(normalized induction)} \tag{1.17}$$

of $G(\mathbb{A})$. It is important to note that the map

$$\varphi \mapsto \Phi(s_0) \tag{1.18}$$

defines a $G(\mathbb{A})$ *intertwining map*:

$$S(V(\mathbb{A})^n) \longrightarrow I_n(s_0,\chi_V). \tag{1.19}$$

This is not true for other values of s and explains why the point $s = s_0$ is the critical one in what follows. Moreover, this map is $H(\mathbb{A})$-invariant, i.e., $\omega(h)\varphi$ and φ have the same image. These two facts will play a key role.

Now let χ be any unitary character of $F_{\mathbb{A}}^{\times}/F^{\times}$ and let $I_n(s,\chi)$ be the induced representation defined with χ in place of χ_V. Note that $I_n(s,\chi)$ is a (restricted) tensor product $I_n(s,\chi) = \otimes_v I_{n,v}(s,\chi_v)$ of the corresponding local induced representations. Note that any $\Phi(s) \in I_n(s,\chi)$ is determined by its restriction to K. A section $\Phi(s) \in I_n(s,\chi)$ – see [3] for an explanation of this terminology – will be called *standard* if its restriction to K is independent of s and *factorizable* if it has the form $\Phi(s) = \otimes_v \Phi_v(s)$ with $\Phi_v(s) \in I_{n,v}(s,\chi_v)$.

For any $\Phi(s) \in I_n(s,\chi)$ and $g \in G(\mathbb{A})$, the Siegel Eisenstein series is defined by:

$$E(g,s,\Phi) = \sum_{\gamma \in P(F)\backslash G(F)} \Phi(\gamma g, s), \tag{1.20}$$

which converges absolutely for $\operatorname{Re} s > \rho_n$, provided $\Phi(s)$ is standard, and defines an automorphic form on $G(\mathbb{A})$. The general results of Langlands (cf. [3]) imply that $E(g,s,\Phi)$ has a meromorphic analytic continuation to the whole s-plane and satisfies a functional equation

$$E(g,s,\Phi) = E(g,-s,M(s)\Phi) \tag{1.21}$$

where

$$M(s) : I_n(s,\chi) \longrightarrow I_n(-s,\chi^{-1}) \tag{1.22}$$

is the global intertwining operator defined, for $\operatorname{Re} s > \rho_n$, by

$$M(s)\Phi(g) = \int_{N(\mathbb{A})} \Phi(ng,s)\,dn. \tag{1.23}$$

1.4 Convergent Siegel–Weil formulas

The classical Siegel–Weil formula gives a relation between the two automorphic forms $I(g;\varphi)$ and $E(g,s,\Phi)$ ($\Phi(s)$ coming from φ) associated to φ in the convergent range. Specifically, assume that $s_0 = \frac{m}{2} - \rho_n > \rho_n$ so that $E(g,s,\Phi)$ is absolutely convergent at $s = s_0$. Note that this condition is equivalent to $m > 2n+2$ so that (1.11)(2) above holds and $I(g;\varphi)$ converges absolutely as well.

Theorem 1.1 ([38], and [41]). *Suppose that $\Phi(s)$ is associated to $\varphi \in S(V(\mathbb{A})^n)$ and that $m > 2n+2$. Then*

$$E(g,s_0,\Phi) = I(g;\varphi).$$

Of course, Weil proved such an identity in *much* greater generality in [41] for dual reductive pairs defined by algebras with involutions.

Main Problem *Extend this result beyond the convergent range.*

The first such extension is the following:

Theorem 1.2 ([19], [20], and [35]). *Assume that Weil's convergence condition (1) or (2) holds so that $I(g;\varphi)$ is well-defined for all $\varphi \in S(V(\mathbb{A})^n)$. Suppose that $\Phi(s)$ is associated to $\varphi \in S(V(\mathbb{A})^n)$. Then*

(i) $E(g,s,\Phi)$ is holomorphic at $s = s_0$;

(ii) and

$$E(g,s_0,\Phi) = \kappa \cdot I(g;\varphi) \qquad \kappa = \begin{cases} 1 & \text{if } m > n+1 \\ 2 & \text{if } m \le n+1. \end{cases}$$

Of course, a number of special cases were known earlier. For example, Siegel himself considered the case $n = 1$, $m = 4$ [38, I, no. 22, p. 443 and III, no. 58].

Remark 1.3. (1) The Eisenstein series is *essentially local* in nature, i.e., it is built out of the local components

$$\Phi_v(g, s) = \omega_v(g)\varphi_v(0)|a(g)|_v^{s-s_0}$$

associated to the local components of a factorizable $\varphi = \otimes_v \varphi_v$. These components can be varied independently and carry no *global* information about V. On the other hand, $I(g; \varphi)$ is *essentially global* in nature, since it involves the theta distribution:

$$\varphi \mapsto \sum_{x \in V(F)^n} \varphi(x)$$

and thus the position of lattice points, etc. The identities above thus contain a link between local and global information. A number theorist might be reminded of the Hasse–Weil or motivic L-functions (which are built out of purely local data) and the elaborate conjectures about their special values (which involve global invariants). It may not be unreasonable to hope that the Siegel–Weil formula may contain information about such special values in certain cases.

(2) Obviously, the Siegel–Weil formula can be viewed either as (i) identifying the theta integral $I(g; \varphi)$ as an Eisenstein series, or (ii) identifying the special value at $s = s_0$ of $E(g, s, \Phi)$ as a theta function, provided the defining data $\Phi(s)$ is associated to φ. For many applications (ii) is a better point of view. It suggests that we should consider the whole family of Eisenstein series $E(g, s, \Phi)$ for *arbitrary* data $\Phi(s) \in I_n(s, \chi)$. This point of view is particularly important outside of the convergent range and is more convenient from a representation-theoretic standpoint.

2

2.1 Some local representation theory

We now consider some of the representation theory which lies behind the Siegel–Weil formula.

We first fix a non-archimedean place v of F and temporarily shift notation, writing F for the completion of the old F at v, and similarly writing $G = \mathrm{Sp}_n(F)$, $H = \mathrm{O}(V)$, etc.

We have a local Weil representation ω of $G \times H$ on the space $S(V^n)$ of Schwartz–Bruhat functions on V^n. The local analogue of the map (1.19) is given by

$$S(V^n) \longrightarrow I_n(s_0, \chi_V) \tag{2.1}$$
$$\varphi \mapsto \Phi(s_0), \tag{2.2}$$

where

$$\Phi(g,s) = \omega(g)\varphi(0) \cdot |a(g)|^{s-s_0}.$$

The map (2.1) is G-intertwining and H-invariant. In fact:

Theorem 2.1 ([34]). *Let*

$$R_n(V) = S(V^n)_H$$

be the maximal quotient of $S(V^n)$ on which H acts trivially. Then the map (2.1) induces an injection of $R_n(V)$ into $I_n(s_0, \chi_V)$:

$$\begin{array}{ccc} S(V^n) & & \\ \downarrow & \searrow & \\ R_n(V) := S(V^n)_H & \hookrightarrow & I_n(s_0, \chi_V). \end{array} \tag{2.3}$$

Thus $R_n(V)$ may be identified with a submodule of $I_n(s_0, \chi_V)$.

Recall that the non-degenerate quadratic spaces over F are classified by the invariants: $\dim_F V = m$, $\det V \in F^\times/F^{\times,2}$ and $\epsilon(V)$, the Hasse invariant of V. Note that $\det V$ and the quadratic character χ_V determine each other uniquely. Thus, if we view $s_0 = \frac{m}{2} - \rho_n$ and $\chi = \chi_V$ as fixed, and if $m > 2$, there are precisely two corresponding quadratic spaces $V_\pm$, having opposite Hasse invariants. When $m = 2$ there are again two spaces $V_\pm$ if $\chi \neq 1$, while, if $\chi = 1$ there is only the split binary space V_ϵ, where $\epsilon = (-1,-1)_F$. Here $(\ ,\ )_F$ is the quadratic Hilbert symbol for F. The induced representation $I_n(s_0, \chi)$ thus has two (resp. one, if $m = 2$ and $\chi = 1$) G-submodules, $R_n(V_\pm)$. It turns out that these submodules account for all of the reducibility of $I_n(s_0, \chi)$. To state the precise result, it is convenient to fix a generator ϖ for the maximal ideal of the ring of integers $\mathcal{O}$ of F and to call a character χ of $F^\times$ *normalized* if $\chi(\varpi) = 1$.

Theorem 2.2 ([12], and [23]). *Assume that χ is a normalized character of $F^\times$. Then*

1. *If $\chi^2 \neq 1$, then $I_n(s,\chi)$ is irreducible for all s.*
2. *If $\chi^2 = 1$ and $\chi \neq 1$, then $I_n(s,\chi)$ is irreducible except at the points s in the set*
$$\frac{m}{2} - \rho_n + \frac{i\pi}{\log q}\mathbb{Z} \qquad 2 \leq m \leq 2n,\ m \text{ even}.$$
3. *If $\chi = 1$, then $I_n(s,\chi)$ is irreducible except at the points s in the set*
$$\frac{m}{2} - \rho_n + \frac{i\pi}{\log q}\mathbb{Z} \qquad 2 \leq m \leq 2n,\ m \text{ even}$$
and at the points $s = \pm\rho_n + \frac{2i\pi}{\log q}\mathbb{Z}$ corresponding to $m = 0$ or $2n + 2$.

Here q is the order of the residue class field of F.

Such induced representations were also considered in [15].

Note that, in cases (2) and (3), the vertical translation by $\frac{i\pi}{\log q}$ should be thought of as a shift of χ to the quadratic character $\chi\cdot|\,|^{\frac{i\pi}{\log q}}$. Thus, in the rest of our discussion, we will consider only quadratic characters χ, *without* the assumption that χ is normalized, and will describe the submodule structure of $I_n(s,\chi)$ at a point $s_0 = \frac{m}{2} - \rho_n$ with $0 \le m \le 2n+2$, m even.

The simplest case occurs when n is odd and $m = n+1$, since the representation $I_n(0,\chi)$ is unitarizable and hence completely reducible. In this case, if $n > 1$, or if $n = 1$ and $\chi \neq 1$,

$$I_n(0,\chi) = R_n(V_+) \oplus R_n(V_-)$$

and the representations $R_n(V_\pm)$ are irreducible. In the case $n = 1$ and $\chi = 1$,

$$I_1(0,1) = R_n(V_\epsilon)$$

is irreducible.

If $2 \le m < n+1$, the representations $R_n(V)$ are irreducible and distinct. In this range, excluding the case $m = 2$ and $\chi = 1$, the subspace $R_n(V_+)\oplus R_n(V_-)$ is the maximal proper submodule of $I_n(s_0,\chi)$. If $m = 2$ and $\chi = 1$, then $R_n(V)$ for the split binary space V is the unique submodule of $I_n(s_0,1)$. Similarly, if $m = 0$ and $\chi = 1$, so that $s_0 = -\rho_n$, the trivial representation $\mathbb{1} = R_n(0)$ is the unique submodule of $I_n(-\rho_n,1)$.

On the right-hand side of the unitary axis, i.e., in the range $n+1 < m \le 2n+2$, the situation is reversed. First of all, excluding the cases $\chi = 1$ and $m = 2n$ or $2n+2$, each of the spaces $V_\pm$ can be written as an orthogonal direct sum

$$V_\pm = V_{0,\pm} + V_{r,r} \tag{2.4}$$

where $V_{0,\pm}$ is a quadratic space with the same Hasse invariant and determinant as $V_\pm$ and of dimension $m_0 = \dim_F V_{0,\pm}$ with

$$m + m_0 = 2n+2. \tag{2.5}$$

Here $V_{r,r}$ is a split space of dimension $2r$, i.e., an orthogonal direct sum of r hyperbolic planes.

Definition 2.3. *The spaces $V_\pm$ and $V_{0,\pm}$ related in this way will be called complementary.*

Note that $R_n(V_{0,\pm}) \subset I_n(-s_0,\chi)$. Then $R_n(V_\pm)$ has a unique irreducible quotient $R_n(V_{0,\pm})$, and the kernel of the map

$$R_n(V_\pm) \longrightarrow R_n(V_{0,\pm}) \tag{2.6}$$

is irreducible [23]. The two submodules $R_n(V_+)$ and $R_n(V_-)$ span $I_n(s_0,\chi)$ and intersect in this irreducible submodule.

In the excluded cases $\chi = 1$ and $m = 2n$ or $2n+2$, the split space has Hasse invariant $\epsilon := (-1,-1)_F^{\frac{n(n+1)}{2}}$ if $m = 2n$ or $(-1,-1)_F^{\frac{(n+1)(n+2)}{2}}$ if $m = 2n+2$. Again we have

$$V_\epsilon = V_{0,\epsilon} + V_{r,r} \tag{2.7}$$

and $R_n(V_\epsilon)$ has a unique irreducible quotient $R_n(V_{0,\epsilon})$. Now, however, $R_n(V_{-\epsilon})$ is the irreducible submodule of $R_n(V_\epsilon)$ which is the kernel of the map

$$R_n(V_\epsilon) \longrightarrow R_n(V_{0,\epsilon}). \tag{2.8}$$

It is important to note that the representations $I_n(s_0,\chi)$ and $I_n(-s_0,\chi)$ are contragradient [34] and are, moreover, related by the normalized local intertwining operator

$$M^*(s_0) : I_n(s_0,\chi) \longrightarrow I_n(-s_0,\chi).$$

This operator is a 'normalized' version of the global intertwining operator defined by (1.23),

$$M^*(s) = \frac{1}{a_n(s,\chi)} M(s), \tag{2.9}$$

where

$$a_n(s,\chi) = L(s+\rho_n - n,\chi) \cdot \prod_{k=1}^{[\frac{n}{2}]} L(2s-n+2k,\chi^2). \tag{2.10}$$

Here $L(s,\chi) = (1-\chi(\varpi)q^{-s})^{-1}$ is the local Euler factor when χ is unramified and is 1 if χ is ramified. In particular, the maps (2.6) and (2.8) can be realized as the restrictions of $M^*(s_0)$ to the subspace $R_n(V_\pm)$ in [23].

Two additional facts will be relevant [23].

1. If V is a split space, then the representation $R_n(V) \subset I_n(s_0,\chi_V)$ contains the normalized 'spherical' vector $\Phi^0(s_0)$ defined by requiring that

$$\Phi^0(k,s_0) = 1. \tag{2.11}$$

2. If $m > 2n+2$, then $R_n(V_\pm) = I_n(s_0,\chi_V)$.

Next we turn to the case of a real archimedean place v, which we again omit from the notation. We let $\mathfrak{g}$ (resp. $\mathfrak{h}$) denote the Lie algebra of $G = \mathrm{Sp}(n,\mathbb{R})$ (resp. $H = \mathrm{O}(V)$). Suppose that $\mathrm{sig}V = (p,q)$ with $p+q = m$, and fix an orthogonal decomposition

$$V = U_+ + U_- \tag{2.12}$$

where $\dim U_+ = p$, $\dim U_- = q$, and $(\ ,\ )|_{U_+}$ (resp. $(\ ,\ )|_{U_-}$) is positive (resp. negative) definite. Then $H \simeq \mathrm{O}(p,q)$, and we let $K_H \simeq \mathrm{O}(p) \times \mathrm{O}(q)$ be the maximal compact subgroup of H which preserves the decomposition (2.12). Let $\mathcal{S}(V^n)$ be the Schwartz space of V^n and define the Gaussian $\varphi^0 \in \mathcal{S}(V^n)$ by $\varphi^0(x) = \exp(-\pi(x,x)_+)$, where $(\ ,\ )_+|_{U_+} = (\ ,\ )$ and $(\ ,\ )_+|_{U_-} =$

$-(\ ,\ )$. Then let $S(V^n) \subset \mathcal{S}(V^n)$ be the space of all functions of the form $\varphi(x) = p(x)\varphi^0(x)$ where p is a polynomial on V^n. The Weil representation of $G \times H$ can be realized on $\mathcal{S}(V^n)$ and $S(V^n)$ then becomes a $(\mathfrak{g}, K) \times (\mathfrak{h}, K_H)$ module (Harish-Chandra module). Note that $S(V^n)$ consists of $(K \times K_H)$-finite functions. Similarly, we let $I_n(s, \chi)$ denote the space of smooth K-finite functions on G which satisfy (1.16).

Theorem 2.4 ([21]). *Let $R_n(p,q)$ be the maximal quotient of $S(V^n)$ on which $(\mathfrak{h}, K_H)$ acts trivially. Then the map* (2.1) *induces an injection of $R_n(p,q)$ into $I_n(s_0, \chi_V)$:*

$$\begin{array}{ccc} & S(V^n) & \\ & \downarrow \qquad \searrow & \\ R_n(p,q) = & S(V^n)_H \hookrightarrow & I_n(s_0, \chi_V). \end{array} \tag{2.13}$$

Thus $R_n(p,q)$ may be identified with a submodule of $I_n(s_0, \chi_V)$.

Note that then

$$\chi_V(x) = (x, (-1)^{\frac{m}{2}+q})_{\mathbb{R}}.$$

Thus, if a quadratic character χ ($\chi(x) = (\mathrm{sgn} x)^a$ with $a = 0$ or 1) and a point $s_0 = \frac{m}{2} - \rho_n$ are fixed, then each pair (p,q) with $p + q = m$ and with $\chi(-1) = (-1)^{\frac{p-q}{2}}$ determines a submodule $R_n(p,q)$ of $I_n(s_0, \chi)$.

The reducibility and submodule structure of $I_n(s, \chi)$ is now considerably more complicated than in the p-adic case. The reducibility, for example, is competely determined in [21]. Here we simply record certain useful facts.

1. The $(\mathfrak{g}, K)$-module $R_n(p,q)$ is generated by the vector $\Phi^\ell(s_0)$ whose value on $K \simeq U(n)$ is given by
$$\Phi^\ell(k, s) = (\det k)^\ell, \tag{2.14}$$
where $\ell = \frac{1}{2}(p - q)$.
2. $R_n(p,q)$ is irreducible if $p + q = m \le n + 1$.
3. On the unitary axis we have
$$I_n(0, \chi) = \bigoplus_{\substack{p+q=m \\ \chi(-1)=(-1)^{\frac{p-q}{2}}}} R_n(p,q). \tag{2.15}$$
4. $R_n(m, 0)$ (resp. $R_n(0, m)$) is an irreducible lowest (resp. highest) weight representation of $(\mathfrak{g}, K)$.

We omit the case of a complex archimedean place. At the moment, we do not know the structure of $I_n(s, \chi)$, although it is undoubtedly known to the experts.

Remark 2.5. The representation $R_n(V)$ always has a unique irreducible quotient, as it must by the local Howe duality principle ([14], [29], and [39]). This quotient is the representation $\theta(\mathbb{1})$ associated to the trivial representation of $H = \mathrm{O}(V)$ via the local theta correspondence.

2.2 Some automorphic representations

We return to the global situation and assemble the local pieces. We assume that χ is a quadratic character of $F_{\mathbb{A}}^{\times}/F^{\times}$. Note that the global induced representation has a factorization:

$$I_n(s,\chi) = \otimes_v I_{n,v}(s,\chi_v) \tag{2.16}$$

as a representation of $(\mathfrak{g}, K_\infty) \times G(\mathbb{A}_f)$. Here we have taken $I_n(s,\chi)$ to consist of smooth, K-finite functions on $G(\mathbb{A})$, and the tensor product on the right side is the restricted tensor product with respect to the vectors $\Phi_v^0(s)$ defined by (2.11). For certain values of s and χ the representation $I_n(s,\chi)$ has infinitely many constituents. For example, if V is a quadratic space over F, as in Section 1 above, if v is non-archimedean (resp. $(\mathfrak{g}_v, K_v)$, if v is archimedean) period and defined in Section 2.1, we let $R_{n,v}(V) = R_n(V_v)$ be the representation of G_v. Since V_v is split at almost all places, we may define the restricted tensor product

$$\Pi_n(V) = \otimes_v R_{n,v}(V), \tag{2.17}$$

and this representation is a submodule of $I_n(s_0, \chi_V)$.

Examples 2.6. 1. If $m \le n+1$, then $\Pi_n(V)$ is an irreducible submodule of $I_n(s_0,\chi_V)$. Note that $s_0 \le 0$.

2. If $m > 2n+2$, then

$$I_{n,f}(s,\chi) = \otimes_{v \text{ finite}} I_{n,v}(s,\chi_v)$$

is irreducible for all s, while $I_{n,\infty}(s,\chi)$ is irreducible unless $s = s_0 = \frac{m}{2} - \rho_n$ for some $m \in 2\mathbb{Z}$. At such points s_0, $I_{n,\infty}(s_0,\chi)$ and hence $I_n(s_0,\chi)$ has finite length.

3. In the range $n+1 < m \le 2n+2$, $I_n(s_0,\chi)$ has infinite length and has no irreducible admissible submodules.

In fact, not every submodule of $I_n(s_0,\chi)$ is associated to a quadratic space. The key examples are defined as follows.

Definition 2.7. *Fix a quadratic character $\chi = \otimes_v \chi_v$ and an even integer $m \ge 0$. A collection $\mathcal{C} = \{U_v\}$ of quadratic spaces U_v over F_v, one for each place v, will be called an* incoherent family *if*

(i) For all places v, $\dim_{F_v} U_v = m$ and $\chi_{U_v} = \chi_v$.
(ii) For almost all v, U_v is the split space of dimension m.
(iii)

$$\prod_v \epsilon_v(U_v) = -1,$$

where $\epsilon_v(U_v)$ is the Hasse invariant of U_v.

In particular, there cannot be a global quadratic space V over F whose localizations are isomorphic to the U_v's since condition (iii) violates the product formula for the Hasse invariants of such localizations.

Note that, if we exclude the case $m = 2$, then an incoherent family $\mathcal{C} = \{U_v\}$ when modified at any one place v (by switching U_v with the space U'_v with the opposite Hasse invariant) becomes the set of localizations $\{V_v\}$ of a global quadratic space V of dimension m and character $\chi_V = \chi$. No particular place v is 'preferred'. In the case $m = 2$ with $\chi \neq 1$ we can only modify at the places v at which $\chi_v \neq 1$. When $m = 2$ and $\chi = 1$ there are no incoherent families.

If $\mathcal{C}$ is an incoherent family, we can form, thanks to (ii), the representation

$$\Pi_n(\mathcal{C}) = \otimes_v R_n(U_v), \tag{2.18}$$

and when $m \leq n + 1$, this representation is an irreducible submodule of $I_n(s_0, \chi)$. In fact, in the range $-\rho_n \leq s_0 \leq n+1$, every irreducible submodule is either a $\Pi_n(V)$ or a $\Pi_n(\mathcal{C})$. Note that we could formally include the case $m = 0$ and obtain $\Pi_n(0) = \mathbb{1}$, the trivial representation.

Now since χ is automorphic, i.e., is a character of $F_{\mathbb{A}}^\times/F^\times$ and not just of $F_{\mathbb{A}}^\times$, it follows from a very general result of Langlands [26] that every irreducible constituent of $I_n(s, \chi)$ occurs as a subquotient of the space of automorphic forms on $G(\mathbb{A})$. This result is proved using the general theory of Eisenstein series and their derivatives. For the representations $\Pi_n(V)$ and $\Pi_n(\mathcal{C})$ we can give much more explicit information.

For example, fix a global quadratic space V and assume that Weil's convergence condition (1.11) holds. Then the theta integral (1.10) defines an intertwining map for the $(\mathfrak{g}, K) \times G(\mathbb{A}_f)$ action,

$$\begin{array}{ccc} S(V(\mathbb{A})^n) & & \\ \downarrow & \overset{I}{\searrow} & \\ \bar{I} : \Pi_n(V) & \longrightarrow & \mathcal{A}(G), \end{array} \tag{2.19}$$

where $\mathcal{A}(G)$ is the space of automorphic forms on $G(\mathbb{A})$. Note that I factors through $\Pi_n(V)$ since it is $H(\mathbb{A})$-invariant.

On the other hand, for $s_0 > \rho_n$, i.e., for $m > 2n + 2$, the Siegel Eisenstein series is termwise absolutely convergent for all $\Phi(s) \in I_n(s, \chi)$, and hence defines an intertwining map

$$\Pi_n(V) \subset I_n(s_0, \chi) \overset{E(s_0)}{\hookrightarrow} \mathcal{A}(G). \tag{2.20}$$

Remark 2.8. The injectivity of $E(s_0)$ can be proved by considering the constant term of $E(g, s_0, \Phi)$ as in [19]. The 'exponents' of the $n+1$ terms in this constant term are distinct in the range $\mathrm{Re}(s) > \rho_n$ so that there can be no cancellations among them. On the other hand, the first term is just $\Phi(g, s)$ itself.

The classical Siegel–Weil formula says that the two maps $\bar{I}$ and $E(s_0)$ agree on $\Pi_n(V)$! Thus, from a representation-theoretic viewpoint, the classical Siegel–Weil formula expresses an *Eisenstein intertwining map* in terms of a *theta intertwining map*. This interpretation has a nice extension to the divergent range.

Remark 2.9. Note: it follows that $\bar{I}$ is injective on $\Pi_n(V)$.

We next suppose that $m \le n$, so that $\Pi_n(V)$ and $\Pi_n(\mathcal{C})$ are irreducible.

Theorem 2.10 ([24]).

(i) In the range $0 \le m \le n$,

$$\dim \operatorname{Hom}_{G(\mathbb{A})}(\Pi_n(V), \mathcal{A}(G)) = 1.$$

Moreover, excluding the case $m = 2$ *and* $\chi = 1$, *the representations* $\Pi_n(V)$ *are square integrable, i.e., occur as invariant subspaces in* $\mathcal{A}_{(2)}(G)$, *the space of square integrable automorphic forms.*

(ii) If $2 \le m \le n+1$ *and* $\mathcal{C}$ *is an incoherent family, then*

$$\operatorname{Hom}_{G(\mathbb{A})}(\Pi_n(\mathcal{C}), \mathcal{A}(G)) = 0,$$

i.e., the irreducible representation $\Pi_n(\mathcal{C})$ *does not occur as an invariant subspace of* $\mathcal{A}(G)$. *It only occurs as a subquotient.*

This uniqueness result is the key to the Siegel–Weil formula in the divergent range.

3

3.1 Poles of the Siegel Eisenstein series

The Siegel Eisenstein series defined using a section $\Phi(s)$ associated to a function $\varphi \in S(V(\mathbb{A})^n)$ where V satisfies Weil's convergence condition (1.11) is always holomorphic at the point $s_0 = \frac{m}{2} - \rho_n$. On the other hand, if $\Phi(s)$ is an arbitrary section and if $0 \le m \le 2n+2$, $E(g, s, \Phi)$ can indeed have a pole. We can give a rather complete description of these poles.

The first step is to normalize the Eisenstein series. Let $\Phi(s) = \otimes_v \Phi_v(s) \in I_n(s, \chi)$ be a factorizable standard section cf. (1.3), and let S be a finite set of places, including all of the archimedean places, such that, for any $v \notin S$, $\Phi_v(s)$ is the normalized spherical vector of (2.11). For any place v of F, let

$$b_{n,v}(s, \chi_v) = L_v(s + \rho_n, \chi_v) \cdot \prod_{k=1}^{[\frac{n}{2}]} L_v(2s + n + 1 - 2k, \chi_v^2), \tag{3.1}$$

where $L_v(s,\chi_v)$ is the local L-factor of χ_v, and let

$$b_n^S(s,\chi) = \prod_{v\notin S} b_{n,v}(s,\chi_v). \tag{3.2}$$

The normalized Eisenstein series is then

$$E^*(g,s,\Phi) = b_n^S(s,\chi)E(g,s,\Phi). \tag{3.3}$$

Theorem 3.1 ([22], and [24]). *Let $\Phi(s)$ be a standard section of $I_n(s,\chi)$, and let S be as above.*

(i) If $\chi^2 \neq 1$, then $E^*(g,s,\Phi)$ is entire.
(ii) If $\chi^2 = 1$, then $E^*(g,s,\Phi)$ has at most simple poles, and these can only occur at the points $s_0 \in X_n^+ = X_n \cap \mathbb{R}_{>0}$, where

$$X_n = \{\ -\rho_n, 1-\rho_n, \dots \rho_n - 1, \rho_n\ \}.$$

Moreover, if $\chi \neq 1$, then no pole occurs at ρ_n.

Note that the normalizing factor $b^S(s,\chi)$ has no poles or zeroes in the right-half plane, so that the behavior of $E(g,s,\Phi)$ in this half plane is the same as that of $E^*(g,s,\Phi)$.

To determine the residues, we fix a quadratic character χ and a point $s_0 = \frac{m}{2} - \rho_n \in X_n^+$. The map

$$A_{-1} : I_n(s_0,\chi) \longrightarrow \mathcal{A}(G) \tag{3.4}$$

$$\Phi(s_0) \mapsto \mathrm{Res}_{s=s_0} E(g,s,\Phi), \tag{3.5}$$

where $\Phi(s) \in I_n(s,\chi)$ is the standard extension of $\Phi(s_0) \in I_n(s_0,\chi)$, defines a $(\mathfrak{g}, K_\infty) \times G(\mathbb{A}_f)$ intertwining operator. Since for each place v the local induced representation $I_{n,v}(s_0,\chi)$ is spanned by the $R_n(U_v)$'s for $\dim_{F_v} U_v = m$ and $\chi_{U_v} = \chi_v$, we may as well assume that our section $\Phi(s) = \otimes_v \Phi_v(s)$ with $\Phi_v(s_0) \in R_n(U_v)$ for each v. Such a section $\Phi(s_0) \in \Pi_n(\{U_v\})$ is called *homogeneous*; the collection of U_v's need not be unique. There are then two cases.

1. The section $\Phi(s_0)$ is 'coherent', i.e., $\Phi(s_0) \in \Pi_n(V)$ for some global quadratic space V over F.
2. The section $\Phi(s_0)$ is incoherent, i.e., the only possible collections $\mathcal{C} = \{U_v\}$ such that $\Phi(s_0) \in \Pi_n(\mathcal{C})$ are incoherent families.

The resulting Eisenstein series will be called 'coherent' and 'incoherent' at the point s_0 respectively.

First consider the case of coherent sections associated to a quadratic space V. If Weil's condition (1.11) is satisfied, then $E(g,s,\Phi)$ is holomorphic at s_0 so the subspace $\Pi_n(V)$ lies in the kernel of the map A_{-1}. If, on the other hand, V is isotropic with $m - r \leq n+1$ where r is the Witt index of V, then (and only then!) there is an orthogonal decomposition

$$V = V_0 + V_{t,t} \tag{3.6}$$

where

$$\dim_F V + \dim_F V_0 = 2n + 2 \tag{3.7}$$

and $V_{t,t}$ is a split space of dimension $2t$. Note that the localizations V_v and $V_{0,v}$ will then be complementary in the sense defined in Section 2. In particular, there is a quotient map

$$\Pi_n(V) \longrightarrow \Pi_n(V_0) \tag{3.8}$$

given by the tensor product of the local quotient maps (2.6) and (2.8), and their archimedean analogue.

Remark 3.2. Since the existence of a complementary V_0 to V is equivalent to the failure of Weil's convergence criterion (1.11) for V, we obtain a structural explanation of that criterion.

Theorem 3.3 ([24]). *Asume that $m = \dim V > n+1$.*

(i) If V satisfies the condition (1.11), then the restriction of A_{-1} to $\Pi_n(V)$ is identically zero [19].
(ii) If V is isotropic with $m - r \leq n+1$, then the restriction of A_{-1} to $\Pi_n(V)$ factors through the quotient map (3.8) and induces an injection

$$\bar{A}_{-1} : \Pi_n(V_0) \hookrightarrow \mathcal{A}(G). \tag{3.9}$$

(iii) The restriction of A_{-1} to $\Pi_n(\mathcal{C})$ for an incoherent family $\mathcal{C}$ is identically zero.

Finally, we consider the center of the critical strip when n is odd. Now $E^*(g, s, \Phi)$ is holomorphic at $s = s_0 = 0$, for any standard section $\Phi(s)$. If we exclude the case $n = 1$ and $\chi = 1$, then the normalizing factor $b_n^S(s, \chi)$ is holomorphic and non-zero at $s = 0$, so that we could just as well consider $E(g, s, \Phi)$ itself. Note that

$$I_n(0, \chi) = \left(\bigoplus_V \Pi_n(V) \right) \oplus \left(\bigoplus_{\mathcal{C}} \Pi_n(\mathcal{C}) \right) \tag{3.10}$$

where V runs over all (isomorphism classes of) quadratic spaces over F of dimension $m = n+1$ and character $\chi_V = \chi$, and $\mathcal{C}$ runs over all incoherent families of the same type.

Theorem 3.4 ([24]). *The map $A_0 : I_n(0, \chi) \longrightarrow \mathcal{A}(G)$ given by*

$$A_0(g, \Phi) = E(g, 0, \Phi)$$

induces an injection

$$\bar{A}_0 : \bigoplus \Pi_n(V) \hookrightarrow \mathcal{A}(G). \tag{3.11}$$

Here $\dim V = m = n+1$. *Moreover the restriction of A_0 to $\bigoplus \Pi_n(\mathcal{C})$ is zero.*

Thus the residues (resp. values) of the Siegel Eisenstein series provide non-zero embeddings of the irreducible $\Pi_n(V)$'s, $\dim V \leq n$ (resp. $\dim V = n+1$ when n is odd) into $\mathcal{A}(G)$. However, this fact is not yet a Siegel–Weil formula. To obtain one we must next realize the $\Pi_n(V)$'s as spaces of theta functions.

3.2 Regularized theta integrals

Let V be an isotropic quadratic space over F, with Witt index r, such that $m - r \le n + 1$ the theta integral (1.10) may thus be divergent. We want to define a regularized version of this integral by using an old trick of Maass [28]: we apply a central differential operator to eliminate the 'bad terms' of the integrand. Similar operators were also found independently by Deitmar and Kreig [6].

Write

$$V = V_{an} + V_{r,r} \tag{3.12}$$

where V_{an} is anisotropic, and assume that $\dim_F V = m \le 2n$.

Recall that W was the standard symplectic space over F which defined $G = \mathrm{Sp}(W) = \mathrm{Sp}(n)$. Then there is another model of the Weil representation of $G(\mathbb{A}) \times H(\mathbb{A})$ on the space $S(V_{an}(\mathbb{A})^n) \otimes S(W(\mathbb{A})^r)$, and the two models are related by a partial Fourier transform:

$$S(V(\mathbb{A})^n) \xrightarrow{\sim} S(V_{an}(\mathbb{A})^n) \otimes S(W(\mathbb{A})^r) \qquad\qquad \varphi \mapsto \hat{\varphi}. \tag{3.13}$$

Poisson summation implies that

$$\sum_{x \in V(F)^n} \varphi(x) = \sum_{\substack{y \in V_{an}(F)^n \\ w \in W(F)^r}} \hat{\varphi}(y, w). \tag{3.14}$$

Thus

$$\theta(g, h; \varphi) = \sum_{\substack{y \in V_{an}(F)^n \\ w \in W(F)^r}} \hat{\omega}(g, h)\hat{\varphi}(y, w). \tag{3.15}$$

It turns out that the terms in this second expression for $\theta(g, h; \varphi)$ which do *not* decay well on $H(F)\backslash H(\mathbb{A})$ are precisely those involving pairs (y, w) with $w \in W(F)^r \simeq M_{r,2n}(F)$ having rank less than r. On the other hand, the sum of the terms involving w's of rank r is rapidly decreasing on $H(F)\backslash H(\mathbb{A})$.

Theorem 3.5 ([24]). *Fix an archimedean place v of F. Then there exist elements $z \in \mathfrak{z}(\mathfrak{g}_v)$ and $z' \in \mathfrak{z}(\mathfrak{h}_v)$, where $z(\mathfrak{g}_v)$ (resp. $\mathfrak{z}(\mathfrak{h}_v)$) is the center of the enveloping algebra of $\mathfrak{g}_v = \mathrm{Lie}\, G_v$ (resp. $\mathfrak{h}_v = \mathrm{Lie}\, H_v$), such that*

(i) $\omega(z) = \omega(z') \ne 0$.
(ii) For all $\varphi_v \in S(V_{an,v}) \otimes S(W_v^r)$, $\big(\omega(z)\varphi_v\big)(y, w) = 0$ whenever the rank of $w \in W_v^r \simeq M_{r,2n}(F_v)$ is less than r.

Corollary 3.6. *$\theta(g, h; \omega(z)\varphi)$ is rapidly decreasing on $H(F)\backslash H(\mathbb{A})$.*

Then we can consider the integral

$$\int_{H(F)\backslash H(\mathbb{A})} \theta(g, h; \omega(z)\varphi)\, E(h, s')\, dh, \tag{3.16}$$

where $E(h,s')$ is the Eisenstein series on $H(\mathbb{A})$ associated to the parabolic subgroup which stabilizes a maximal isotropic subspace of V, and normalized so that $E(h,s')$ has constant residue 1 at the point $s' = s_0' = \rho'$, analogous to ρ_n. This integral is absolutely convergent whenever $E(h,s')$ is holomorphic. For large $\Re(s')$ it can be unfolded in the usual way and turns out to be equal to

$$P(s';z)\cdot \mathcal{E}(g,s',\varphi) \tag{3.17}$$

where $P(s',z)$ is an explicit polynomial in s' and $\mathcal{E}(g,s',\varphi)$ is an Eisenstein series on $G(\mathbb{A})$ associated to a maximal parabolic subgroup P_r of G which stabilizes an isotropic r-plane in W. Then

$$\mathcal{E}(g,s',\varphi) = \frac{1}{P(s';z)}\int_{H(F)\backslash H(\mathbb{A})} \theta(g,h;\omega(z)\varphi)\,E(h,s')\,dh.$$

If φ is such that $\theta(g,h;\varphi)$ is already rapidly decreasing on $H(F)\backslash H(\mathbb{A})$, then

$$\int_{H(F)\backslash H(\mathbb{A})} \theta(g,h;\omega(z)\varphi)\,E(h,s')\,dh \tag{3.18}$$

$$= \int_{H(F)\backslash H(\mathbb{A})} \omega(z')\cdot\theta(g,h;\varphi)\,E(h,s')\,dh \tag{3.19}$$

$$= \int_{H(F)\backslash H(\mathbb{A})} \theta(g,h;\varphi)\ \omega(z')^*\cdot E(h,s')\,dh \tag{3.20}$$

$$= P(s';z)\cdot\int_{H(F)\backslash H(\mathbb{A})} \theta(g,h;\varphi)\,E(h,s')\,dh \tag{3.21}$$

and thus

$$\mathcal{E}(g,s',\varphi) = \int_{H(F)\backslash H(\mathbb{A})} \theta(g,h;\varphi)\,E(h,s')\,dh \tag{3.22}$$

in this case.

3.3 Extended Siegel–Weil formulas

We are interested in the Laurent expansion of $\mathcal{E}(g,s',\varphi)$ at the point s_0'. The polynomial $P(s';z)$ has the property that

$$\operatorname{ord}_{s_0'} P(s';z) = \begin{cases} 0 & \text{if } 2\le m\le n+1 \\ +1 & \text{if } n+1<m\le 2n. \end{cases} \tag{3.23}$$

Thus

$$\operatorname{ord}_{s_0'} \mathcal{E}(g,s',\varphi) = \begin{cases} -1 & \text{if } 2\le m\le n+1 \\ -2 & \text{if } n+1<m\le 2n, \end{cases} \tag{3.24}$$

and we have

$$\mathcal{E}(g,s',\varphi) = \frac{B_{-2}(g,\varphi)}{(s'-s_0')^2} + \frac{B_{-1}(g,\varphi)}{(s'-s_0')} + O(1). \tag{3.25}$$

We view the various terms in this expansion as the regularization of the theta integral (1.10).

Theorem 3.7 ([24]). *Let V be an isotropic quadratic space over F with $m-r \le n+1$.*

(i) If $2 \le m \le n+1$, then the map B_{-1} induces a diagram:

$$\begin{array}{ccc} S(V(\mathbb{A})^n) & & \\ \downarrow & \overset{B_{-1}}{\searrow} & \\ \bar{I} = \bar{B}_{-1} : \Pi_n(V) & \hookrightarrow & \mathcal{A}(G). \end{array} \tag{3.26}$$

(ii) If $n+1 < m \le 2n$, let V_0 be the complementary space to V. Then the map B_{-2} factors through $\Pi_n(V)$ and further factors through the quotient $\Pi_n(V_0)$, i.e.,

$$\begin{array}{ccc} S(V(\mathbb{A})^n) & & \\ \downarrow & \overset{B_{-2}}{\searrow} & \\ \bar{\bar{I}} = \bar{B}_{-2} : \Pi_n(V_0) & \longrightarrow & \mathcal{A}(G). \end{array} \tag{3.27}$$

As a consequence of the uniqueness result of Section 2, we finally obtain a Siegel–Weil formula:

Corollary 3.8 ([24], Siegel–Weil formula for residues and central values).

(i) Assume that $n+1 < m \le 2n$ and let V be an m dimensional isotropic quadratic space over F with $m-r \le n+1$. Then there exists a non-zero constant c_1 such that

$$\bar{A}_{-1} = c_1 \cdot \bar{B}_{-1} = c_1 \cdot \bar{I},$$

where $\bar{A}_{-1}$ is the map on $\Pi_n(V_0)$ induced by the restriction of A_{-1} to $\Pi_n(V)$. Note that $\bar{B}_{-1}$ here is defined as in (3.26) for the complementary space V_0.

(ii) Assume that n is odd and that $n+1=m$ and let V be an m-dimensional quadratic space over F. Then there exists a non-zero constant c_0 such that

$$A_0|_{\Pi_n(V)} = \begin{cases} c_0 \cdot \bar{B}_{-1} = c_0 \cdot \bar{I} & \text{if } V \text{ is isotropic,} \\ 2I & \text{if } V \text{ is anisotropic.} \end{cases}$$

Here the case $n=1$ and $\chi=1$ is excluded.

3.4 Second term identities

One would like to obtain further relations among the terms of the Laurent expansions of the Siegel Eisenstein series $E(g,s,\Phi)$, for $\Phi(s)$ associated to φ, and those of the regularized theta integral $\mathcal{E}(g,s',\varphi)$. Specifically, we would like to consider

$$E(g,s,\Phi) = \frac{A_{-1}(g,\Phi)}{s-s_0} + A_0(g,\Phi) + O(s-s_0), \tag{3.28}$$

and to express A_0 and a combination of B_{-1} and B_{-2}. So far this has only been done in some special cases.

Consider the case $n=2$ and $m=4$, so that $s_0 = \frac{1}{2}$.

Theorem 3.9 ([25]). *There exists a constant c' such that*

$$A_0(g,\Phi) = c'B_{-1}(g,\varphi) + B_{-2}(g,\varphi')$$

where $\Phi(s)$ is the standard section associated to φ and for some function $\varphi' \in S(V(\mathbb{A})^2)$.

Remark 3.10. Such a relation must be rather subtle since the map $\varphi \mapsto A_0(g,\Phi)$ is $H(\mathbb{A})$ invariant but *not*(!!) $G(\mathbb{A})$ intertwining (since the *second* term in the Laurent expansion is not $G(\mathbb{A})$ intertwining (cf. [20, §2]) while the map $\varphi \mapsto B_{-1}(g,\varphi)$ is $G(\mathbb{A})$ intertwining but *not* (!!) $H(\mathbb{A})$ invariant. This explains the presence of the term $B_{-2}(g,\varphi')$ and the occurrence of the additional function φ'. For example, if φ is replaced by $\omega(h)\varphi$, the term $A_0(g,\Phi)$ is unchanged, but the term $B_{-1}(g,\varphi)$ only remains invariant *modulo* the subspace $\mathrm{Im}(B_{-2}) \subset \mathcal{A}(G)$. The convoluted arguments needed in Section 6 of [25] reflect these difficulties.

Remark 3.11. If $\Phi(s)$ is a standard section which is 'incoherent' at the point $s=\frac{1}{2}$, such that $\Phi(\frac{1}{2}) \in \Pi_2(\mathcal{C})$ for some incoherent family $\mathcal{C}$ but does not lie in any $\Pi_n(V)$, then the nature of $A_0(g,\Phi)$ remains to be determined.

A second term identity has also been proved for the case when $V = V_{r,r}$ is a split space and $\varphi \in S(V(\mathbb{A})^n)$ is invariant under K, the maximal compact subgroup of $G(\mathbb{A})$. In this case, the section $\Phi(s)$ associated to φ is (up to a constant which we may assume to be 1) just

$$\Phi(s) = \Phi^0(s) = \otimes_v \Phi_v^0(s), \tag{3.29}$$

where $\Phi_v^0(s)$ is the normalized K_v-invariant standard section of (2.11) or (2.14) with $\ell = 0$. On the other hand, the Eisenstein series $\mathcal{E}(g,s',\varphi)$ is also associated to a K-invariant (but *not* standard) section of an induced representation for the maximal parabolic P_r of G.

More precisely, for any r with $1 \le r \le n$ let P_r be the maximal parabolic subgroup of $G = \mathrm{Sp}(n)$ which stabilizes the isotropic r-plane spanned by the vectors $e'_1, \dots, e'_r$. Here we have fixed a standard symplectic basis $e_1, \dots, e_n, e'_1, \dots, e'_n$ for $W \simeq F^{2n}$ (row vectors), so that $\begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix}$ is the matrix for $\langle\ ,\ \rangle$. Then $P_r = N_r M_r$ where the Levi factor $M_r \simeq \mathrm{GL}(r) \times \mathrm{Sp}(n-r)$. For the global Iwasawa decomposition $G(\mathbb{A}) = P_r(\mathbb{A})K$ we write $g = n_r m_r(a, g_0)k$ with $a \in \mathrm{GL}(r, \mathbb{A})$ and $g_0 \in \mathrm{Sp}(n-r, \mathbb{A})$ and we set

$$|a_r(g)| = |\det(a)|_{\mathbb{A}}. \tag{3.30}$$

Let

$$\mathcal{E}(g, s; r, n) = \sum_{\gamma \in P_r(F)\backslash G(F)} |a_r(\gamma g)|^{s+\rho_{r,n}}, \tag{3.31}$$

where $\rho_{r,n} = n - \frac{r-1}{2}$. We normalize this series by setting

$$\mathcal{E}^*(g, s; r, n) := c_{r,n}(s) b_r(s) \mathcal{E}(g, s; r, n), \tag{3.32}$$

where $b_r(s) = b_r(s, \chi)$ for $\chi = 1$ is defined by (3.1) with $S = \phi$ and

$$c_{r,n}(s) = \prod_{i=0}^{r-1} L(s + \rho_{r,n} - i). \tag{3.33}$$

Here we take the full Euler product, $L(s) = L(s,1) = \prod_v L_v(s,1)$, including the archimedean factors. Similarly, we let

$$E^*(g, s, \Phi^0) = b_n(s,1) E(g, s, \Phi^0) \tag{3.34}$$

be the spherical Siegel Eisenstein series, normalized with the full Euler product $b_n(s, \chi)$ with $\chi = 1$.

From now on we assume that $F = \mathbb{Q}$ so that $L(s) = L(1-s)$ and

$$L(s) = \frac{1}{s-1} + \kappa + O(s-1). \tag{3.35}$$

Note that the functional equation of $E^*(s, s, \Phi^0)$ is then simply

$$E^*(g, s, \Phi^0) = E^*(g, -s, \Phi^0), \tag{3.36}$$

because

$$M(s)\Phi^0 = \frac{a_n(s)}{b_n(s)} \Phi^0(-s) \tag{3.37}$$

and $a_n(s) = b_n(-s)$. By a very elaborate inductive argument one can prove the following:

Theorem 3.12 ([18], first term identities). *Let $s_0 = r - \rho_n = r - \frac{n+1}{2}$.*

(i) If $s_0 < 0$, then

$$\mathrm{Res}_{s=s_0} E^*(g, s, \Phi^0) = c(r,n)\, \mathrm{Res}_{s=\frac{r-1}{2}} \mathcal{E}^*(g, s; r, n).$$

(ii) If $s_0 = 0$, so that n is odd, then

$$E^*(g, 0, \Phi^0) = c(r,n)\,\mathrm{Res}_{s=\frac{r-1}{2}}\,\mathcal{E}^*(g, s; r, n).$$

(iii) If $0 < s_0 < \rho_n$, then

$$\mathrm{Res}_{s=s_0} E^*(g, s, \Phi^0) = c(r,n)\,\mathrm{Res}_{s=\frac{r-1}{2}} \left((s - \frac{r-1}{2})\mathcal{E}^*(g, s; r, n) \right).$$

Here the constant $c(r,n)$ is defined by $c(1,1) = 1$ and

$$c(r, 2r-1)^{-1} = \frac{1}{2}\kappa L(3)L(5)\cdots L(2\left[\frac{r}{2}\right] - 1),$$

for $r > 1$. If $1 \le k \le r-1$, then

$$c(r, 2r-1-k)^{-1} = c(r, 2r-1)^{-1} 2L(2)L(4)\cdots L(2\left[\frac{k}{2}\right]).$$

Finally, for $\ell \ge 1$,

$$c(r, 2r-1+\ell) = -c(r, 2r-1)\frac{1}{2}\kappa L(3)L(5)\cdots L(2\left[\frac{\ell-1}{2}\right] + 1).$$

Theorem 3.13 ([18], spherical second term identity). *Let $s_0 = r - \rho_n = r - \frac{n+1}{2} = \frac{k}{2}$ for $k \in \mathbb{Z} > 0$. Note that $\mathcal{E}^*(g, s; r, n)$ is then associated to $O(r,r) = O(V_{r,r})$ while $\mathcal{E}^*(g, s; r-k, n)$ is associated to the complementary space $O(r-k, r-k) = O(V_{r-k,r-k})$. Let*

$$D(g, s; r, n) = \mathcal{E}^*(g, s; r, n) + \beta(s; r, n)\gamma(s; r, n)\mathcal{E}^*(g, s - \frac{k}{2}; r-k, n),$$

where

$$\beta(s; r, n) = \begin{cases} L(2s)L(2s-2)\cdots L(2s - 2\left[\frac{k-1}{2}\right]) & \text{if } r \text{ is even} \\ L(2s-1)L(2s-3)\cdots L(2s - 2\left[\frac{k}{2}\right] + 1) & \text{if } r \text{ is odd,} \end{cases}$$

and with $\gamma(s; r, n)$ defined inductively by $\gamma(s; r, 2r-1) = 1$ and

$$\gamma(s; r, n) = \gamma(s - \frac{1}{2}; r-1, n-1) \begin{cases} L(s - \frac{r-1}{2} - k + 1) & \text{if } n \text{ is even} \\ L(s - \frac{r-1}{2} + k) & \text{if } n \text{ is odd.} \end{cases}$$

Then $D(g, s; r, n)$ has at most a simple pole at $s = \frac{r-1}{2}$ and, writing

$$E^*(g, s, \Phi^0) = \frac{A_{-1}(g, \Phi^0)}{s - s_0} + A_0(g, \Phi^0) + O(s - s_0),$$

we have

$$A_0(g, \Phi^0) = c(r,n)\,\mathrm{Res}_{s=\frac{r-1}{2}}\, D(g, s; r, n),$$

where $c(r,n)$ is as in Theorem 3.12.

Remark 3.14. The main idea here is that $\mathcal{E}^*(g, s; r, n)$ has a Laurent expansion (3.25), with a second order term which generates a copy of $\Pi_n(V_{r-k,r-k})$ in $\mathcal{A}(G)$, while $\mathcal{E}^*(g, s; r-k, n)$ has only a simple pole at $s = \frac{r-k-1}{2}$, whose residue also generates a copy of $\Pi_n(V_{r-k,r-k})$. The factor $\beta(s; r, n)\gamma(s; r, n)$ also has a simple pole at $s = \frac{r-1}{2}$, and has been taken so that the second order terms cancel in the sum $D(g, s; r, n)$. Miraculously, the given choice exactly expresses the second term in the Laurent expansion of the Siegel Eisenstein series.

Much work remains to be done to establish a second term identity in the general case.

4

4.1 Applications to poles of Langlands L-functions

The rather complete information given in the previous sections about the poles of the Siegel Eisenstein series yields corresponding information about the poles of the standard Langlands L-functions for automorphic representations of the symplectic group.

First recall that the L-group of $G = \mathrm{Sp}(n)$ is ${}^LG = \mathrm{SO}(2n+1, \mathbb{C}) \times W_F$ where W_F is the global Weil group of F. Let $r : {}^LG \longrightarrow \mathrm{GL}(2n+1, \mathbb{C})$ be the representation which is the standard representation on $\mathbb{C}^{2n+1}$ on the first factor and is trivial on W_F. If $\pi \simeq \otimes_v \pi_v$ is an irreducible cuspidal automorphic representation of $G(\mathbb{A})$, then for all places v outside of a finite set $S = S(\pi)$, which contains all of the archimedean places, the local component π_v of π is the spherical constituent of an unramified principal series representation. Such an unramified principal series representation is determined by its Satake parameter $t_v \in {}^LG$.

More precisely, let

$$B = \{nm(a) \in P \mid \text{ such that } a \text{ is upper triangular } \}, \tag{4.1}$$

and write $B = TU$ where U is the unipotent radical and

$$T = \{m(a) \mid a = \text{ diagonal } \}. \tag{4.2}$$

For any non-archimedean place v of F and for any

$$t_v^0 = \mathrm{diag}(q_v^{-\lambda_1}, \ldots, q_v^{-\lambda_n}, 1, q_v^{\lambda_1}, \ldots, q_v^{\lambda_n}) \in \mathrm{SO}(2n+1, \mathbb{C}) \tag{4.3}$$

where $\lambda_j \in \mathbb{C}$, the induced representation $\mathrm{Ind}_{B_v}^{G_v}(\lambda)$ is given by the right multiplication action of G_v on the space of smooth functions f on G_v such that

$$f(um(a)g) = |a_1|_v^{\lambda_1+n} |a_2|_v^{\lambda_2+n-1} \cdots |a_n|_v^{\lambda_n+1} f(g), \tag{4.4}$$

where $a = \mathrm{diag}(a_1, \ldots, a_n) \in \mathrm{GL}(n, F_v)$. Since $G_v = B_v K_v$ this representation has a unique K_v invariant vector f_v^0 determined by the condition $f_v^0(k) = 1$ for all $k \in K_v$. The spherical constituent of $\mathrm{Ind}_{B_v}^{G_v}(\lambda)$ is the unique irreducible constituent containing f_v^0. Let

$$t_v = t_v^0 \times Fr_v \in {}^L G_v = \mathrm{SO}(2n+1, \mathbb{C}) \times W_{F_v}, \tag{4.5}$$

where Fr_v is a Frobenius element of W_{F_v}.

The local Euler factor attached to π_v is then

$$L_v(s, \pi_v, r) = \det(1 - q_v^{-s} r(t_v))^{-1}. \tag{4.6}$$

Note that it has degree $2n+1$. The standard Langlands L-function of π is

$$L^S(s, \pi, r) = \prod_{v \notin S} L_v(s, \pi_v, r). \tag{4.7}$$

A little more generally, if χ is a character of $F_{\mathbb{A}}^\times / F^\times$, let $S = S(\pi, \chi)$ be the union of $S(\pi)$ with the set of non-archimedean places v at which χ is ramified. Then, for $v \notin S$, define the twisted Euler factor

$$L_v(s, \pi_v, \chi_v, r) = \det(1 - q_v^{-s} \chi_v(\varpi_v) r(t_v))^{-1}, \tag{4.8}$$

with ϖ_v a generator of the maximal ideal in the ring of integers $\mathcal{O}_v$ of F_v, and define the L-function

$$L^S(s, \pi, \chi, r) = \prod_{v \notin S} L_v(s, \pi_v, \chi_v, r). \tag{4.9}$$

The meromorphic analytic continuation and functional equation of $L^S(s, \pi, \chi, r)$ can be obtained by the 'doubling method' integral representation of Piatetski-Shapiro and Rallis, [9],[32] and also [4],[8], which we now briefly recall. Let $\widetilde{G} = \mathrm{Sp}(2n)$ and let ι_0 be the embedding $\iota_0 : G \times G \longrightarrow \widetilde{G}$ given by

$$\iota_0(g_1, g_2) = \begin{pmatrix} a_1 & & b_1 & \\ & a_2 & & b_2 \\ c_1 & & d_1 & \\ & c_2 & & d_2 \end{pmatrix}, \tag{4.10}$$

where $g_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix} \in G$. For $g \in G$, let

$$g^\vee = \begin{pmatrix} 1_n & \\ & -1_n \end{pmatrix} g \begin{pmatrix} 1_n & \\ & -1_n \end{pmatrix}, \tag{4.11}$$

and let

$$\iota(g_1, g_2) = \iota_0(g_1, g_2^\vee). \tag{4.12}$$

Also let

$$\delta = \begin{pmatrix} 0 & 0 & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \in \widetilde{G}. \tag{4.13}$$

Choose f_1 and $f_2 \in \pi$ such that f_i is invariant under K_v for all $v \notin S$. Also choose $\Phi(s) = \otimes_v \Phi_v(s) \in I_{2n}(s,\chi)$ such that $\Phi_v(s) = \Phi_v^0(s)$, the normalized $\tilde{K}_v = \mathrm{Sp}(2n, \mathcal{O}_v)$ invariant vector in $I_{2n,v}(s,\chi_v)$ for all $v \notin S$. Let $E^*(g,s,\Phi)$ be the normalized Siegel Eisenstein series on $\widetilde{G}(\mathbb{A})$ and consider the integral

$$Z^*(s,f_1,f_2,\Phi) := \int_{(G(F)\backslash G(\mathbb{A}))\times(G(F)\backslash G(\mathbb{A}))} f_1(g_1)\overline{f_2(g_2)}\, E^*(\iota(g_1,g_2),s,\Phi)\, dg_1\, dg_2. \tag{4.14}$$

The main identity of the doubling method [9] then asserts that

$$Z^*(s,f_1,f_2,\Phi) = L^S(s+\frac{1}{2},\pi,\chi,r)\,\langle \pi_S(\Phi(s))f_1,f_2\rangle, \tag{4.15}$$

where $\Phi_S(s) = \otimes_{v\in S}\Phi_v(s)$ is a function on $G_S = \prod_{v\in S} G_v$, and

$$\langle \pi_S(\Phi(s))f_1,f_2\rangle = \int_{G_S} \langle \pi(g)f_1,f_2\rangle\, \Phi_S(\delta\cdot\iota(g,1),s)\, dg. \tag{4.16}$$

Note that the shift to $s+\frac{1}{2}$ in the L-function corresponds to the fact that our Eisenstein series have $s=0$ as their center of symmetry while the Langlands L-functions have functional equations relating s and $1-s$. From (4.14), it is clear that the poles of $Z^*(s,f_1,f_2,\Phi)$ arise from the poles of $E^*(g,s,\Phi)$. On the other hand, the factor (4.16) arising from the places in the set S can be controlled by making a good choice of the local data $f_{1,v}$, $f_{2,v}$, and $\Phi_v(s)$ at places $v\in S$. Specifically, if v is a non-archimedean place, then the local data can be chosen so that the associated factor in (4.16) is identically 1, while for an archimedean place, the local data can be chosen so that, at any given s_0, the associated factor has neither a zero nor a pole [22]. We can thus apply our previous results, noting that the Eisenstein series in question is on $\widetilde{G}(\mathbb{A})$ rather than on $G(\mathbb{A})$.

We normalize χ by fixing an isomorphism $F_{\mathbb{A}}^\times/F^\times \simeq F_{\mathbb{A}}^1/F^\times \times \mathbb{R}_+^\times$ and assuming that χ is trivial on the $\mathbb{R}_+^\times$ factor. Then we have a precise description of the location of the poles of $L^S(s,\pi,\chi,r)$.

Theorem 4.1 ([24]).

(i) If $\chi^2 \neq 1$, then $L^S(s,\pi,\chi,r)$ is entire.
(ii) If $\chi^2 = 1$, then $L^S(s,\pi,\chi,r)$ has at most simple poles, and these can only occur at the points $s\in\{1,2,\dots,\left[\frac{n}{2}\right]+1\}$.

Next we interpret the poles which occur. Let V be a quadratic space over F of dimension m and character χ_V. Let $H = \mathrm{O}(V)$ as before and, for an irreducible automorphic cuspidal representation π of $G(\mathbb{A})$, as above, let $\Theta(\pi)$ denote the space of automorphic forms on $H(\mathbb{A})$ given by the theta integrals

$$\theta(h; f, \varphi) = \int_{G(F)\backslash G(\mathbb{A})} f(g)\, \theta(g, h; \varphi)\, dg, \tag{4.17}$$

where $f \in \pi \subset \mathcal{A}(G)$ and $\varphi \in S(V(\mathbb{A})^n)$ is K-finite and K_H-finite. Here K_H is some fixed maximal compact subgroup of $H(\mathbb{A})$.

Theorem 4.2 ([24]). *Let π and χ be as before, with $\chi^2 = 1$. Suppose that $L^S(s, \pi, \chi, r)$ has a pole at the point $s = s_0 \in \{1, 2, \ldots, \left[\frac{n}{2}\right] + 1\}$. Let $m = 2n + 2 - 2s_0$. Then there exists a quadratic space V over F with $\dim_F V = m$ and $\chi_V = \chi$ such that $\Theta_V(\pi) \neq 0$.*

Thus the existence of a pole of $L^S(s, \pi, \chi, r)$ at s_0 indicates the non-triviality of a theta lift of π. The further to the right the pole occurs, the smaller the space V. For example, if n is even, the rightmost possible pole occurs at $s_0 = \frac{n}{2} + 1$, and if such a pole does occur, then π has a non-trivial theta lift $\Theta_V(\pi)$ for some space V of dimension n. This result was proved earlier by Piatetski-Shapiro and Rallis [31], using Andrianov's method [1] and [2]. By a result of Li [27], $\Theta_{V'}(\pi)$ must be zero for all quadratic spaces V' of dimension less than n.

The proof of this last result is based on the following:

Proposition 4.3 ([24]). *Let V be a quadratic space over F of dimension m with $m \leq 2n$.*

(i) Suppose that V is anisotropic and let $\bar{I} : \Pi_{2n}(V) \longrightarrow \mathcal{A}(\widetilde{G})$ be the intertwining map given by (2.19). Then, for $\varphi = \varphi_1 \otimes \bar{\varphi}_2 \in S(V(\mathbb{A})^{2n})$,

$$\int_{[G\times G]} f_1(g_1)\overline{f_2(g_2)}\, \bar{I}(\iota(g_1, g_2), \varphi)\, dg_1\, dg_2 = \int_{H(F)\backslash H(\mathbb{A})} \theta(h; f_1, \varphi_1)\overline{\theta(h; f_2 \varphi_2)}\, dh. \tag{4.18}$$

Here $[G \times G] = (G(F)\backslash G(\mathbb{A})) \times G(F)\backslash G(\mathbb{A}))$.

(ii) Suppose that V is isotropic and let $\bar{B}_{-1} : \Pi_{2n}(V) \longrightarrow \mathcal{A}(\widetilde{G})$ be the intertwining map given by (3.26). Then, for $\varphi = \varphi_1 \otimes \bar{\varphi}_2 \in S(V(\mathbb{A})^{2n})$,

$$\int_{[G\times G]} f_1(g_1)\overline{f_2(g_2)}\, \bar{B}_{-1}(\iota(g_1, g_2), \varphi)\, dg_1\, dg_2 = c \int_{H(F)\backslash H(\mathbb{A})} \Big(\theta(h; f_1, \varphi_1)\overline{\theta(h; f_2 \varphi_2)}\Big) * z'\, dh, \tag{4.19}$$

where $z' \in \mathfrak{z}(\mathfrak{h}_v)$ is the element, defined in the Theorem 3.5 of Section 3.2, used to regularize the theta integral.

This result relates the residues of $Z^*(s, f_1, f_2, \Phi)$ to the (regularized) Petersson inner products of theta lifts $\theta(h; f, \varphi)$. This relation – Rallis' inner product formula [33] – was the starting point of the work of Piatetski-Shapiro and Rallis on the 'doubling method'.

4.2 Translation to classical language

It might be useful to explain the relation between the adelic Eisenstein series $E(g, s, \Phi)$ on $G(\mathbb{A})$ and the classical Eisenstein series of Siegel ([38], [37], [16], [7], and [5]). For this it is simplest to work over $F = \mathbb{Q}$.

Note that, by the strong approximation theorem,

$$G(\mathbb{A}) = G(\mathbb{Q})G(\mathbb{R})K' \tag{4.20}$$

where K' is any compact open subgroup of $G(\mathbb{A}_f)$. We will usually take K' to be a subgroup of finite index in $K = \prod_p K_p$ where $K_p = \mathrm{Sp}(n, \mathbb{Z}_p)$. Note that we are slightly changing the notation of the previous sections. Let

$$\Gamma = G(\mathbb{Q}) \cap (G(\mathbb{R})K) = \mathrm{Sp}(n, \mathbb{Z}) \tag{4.21}$$

and let

$$\Gamma' = G(\mathbb{Q}) \cap (G(\mathbb{R})K'). \tag{4.22}$$

Let

$$\Phi(s) = \Phi_\infty(s) \otimes \Phi_f(s) \tag{4.23}$$

with

$$\Phi_f(s) = \otimes_p \Phi_p(s) \tag{4.24}$$

be a standard (restriction to K is independent of s) factorizable section. We assume that $\Phi_f(s)$, the finite part of $\Phi(s)$ is invariant under K' (as it must be for a sufficiently small K'); and thus the Eisenstein series $E(g, s, \Phi)$, which is left $G(\mathbb{Q})$-invariant and right K'-invariant, is determined by its restriction to $G(\mathbb{R})$, which we view as embedded in $G(\mathbb{A})$ via $g_\infty \mapsto (g_\infty, 1, \dots)$.

Note that

$$P(\mathbb{Q})\backslash G(\mathbb{Q}) \simeq (P(\mathbb{Q}) \cap \Gamma)\backslash \Gamma. \tag{4.25}$$

Thus, we have

$$E(g_\infty, s, \Phi) = \sum_{\gamma \in P(\mathbb{Q}) \cap \Gamma \backslash \Gamma} \Phi_\infty(\gamma g_\infty)\Phi_f(\gamma), \tag{4.26}$$

and we must determine the two factors $\Phi_f(\gamma)$ and the function $\Phi_\infty(\gamma g_\infty)$. Note that the function $\Phi_f(\gamma, s)$ depends only on the restriction of $\Phi_f(s)$ to K, and that the data $\Phi_f(s)$ and $\Phi_\infty(s)$ can be varied independently.

Assume that $N \in \mathbb{Z}_{>0}$ is such that $K' = \prod_{p|N} K'_p \times \prod_{p\nmid N} K_p$ and $\Phi_p(s) = \Phi^0_p(s)$, the normalized spherical section of (2.11), for all p prime to N. Then

$$\Phi_f(\gamma) = \prod_{p|N} \Phi_p(\gamma). \tag{4.27}$$

Here the functions $\phi_p = \Phi_p : K'\backslash K \longrightarrow \mathbb{C}$ can be chosen arbitrarily. Nontrivial examples for the case $n = 3$ can be found in Section 3 of [10].

Recall that $G_\infty = \mathrm{Sp}(n, \mathbb{R}) = P(\mathbb{R})K_\infty$ where $K_\infty \simeq \mathrm{U}(n)$ as in Section 1.1. In particular, $\Phi_\infty(s)$ is determined by its restriction to K_∞. The simplest possible choice of $\Phi_\infty(s)$ will be the function determined by

$$\Phi^\ell_\infty(k, s) = (\det \mathbf{k})^\ell \tag{4.28}$$

where $\mathbf{k} \in \mathrm{U}(n)$ corresponds to $k \in \mathrm{Sp}(n, \mathbb{R})$. For this choice of $\Phi_\infty(s)$, the function $g_\infty \mapsto E(g_\infty, s, \Phi)$ is an eigenfunction for the right action of K_∞, so it suffices to describe its values on $P(\mathbb{R})$. Write $g = n(x)m(v) \in P(\mathbb{R})$, and set

$$z = x + iy = x + i\,{}^t vv = g(i \cdot 1_n) \in \mathfrak{H}_n, \tag{4.29}$$

the Siegel space of genus n. Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, and let $\gamma g = nm(\alpha)k$ for the Iwasawa decomposition, with $\alpha \in \mathrm{GL}^+(n, \mathbb{R})$, i.e., with $\det \alpha > 0$. Then we have

$$(0, 1_n) \cdot \gamma g \begin{pmatrix} 1_n \\ i \cdot 1_n \end{pmatrix} = i \cdot {}^t\alpha^{-1}\mathbf{k}. \tag{4.30}$$

On the other hand,

$$(0, 1_n) \cdot \gamma g \begin{pmatrix} 1_n \\ i \cdot 1_n \end{pmatrix} = i \cdot (c\bar{z} + d)^t v^{-1}. \tag{4.31}$$

Consequently

$$\det(\alpha) = \det(v) \cdot |\det(cz + d)|^{-1} \tag{4.32}$$

and

$$\det(\mathbf{k}) = \frac{\det(c\bar{z} + d)}{|\det(cz + d)|}. \tag{4.33}$$

This yields:

$$\Phi^\ell_\infty(\gamma g, s) = \det(v)^{s+\rho_n} |\det(cz + d)|^{-s-\rho_n-\ell} \det(c\bar{z} + d)^\ell \tag{4.34}$$

$$= \det(y)^{\frac{1}{2}(s+\rho_n)} \det(cz + d)^{-\frac{1}{2}(s+\rho_n+\ell)} \det(c\bar{z} + d)^{-\frac{1}{2}(s+\rho_n-\ell)} \tag{4.35}$$

$$= \det(y)^{\frac{1}{2}(s+\rho_n)} \det(cz + d)^{-\ell} |\det(cz + d)|^{-s-\rho_n+\ell}. \tag{4.36}$$

More generally, since $\Phi_\infty(s)$ is standard and K_∞-finite, we may write

$$\Phi_\infty(k, s) = \phi(\mathbf{k}) \tag{4.37}$$

for smooth function ϕ on $\mathrm{U}(n)$ which is left invariant under $\mathrm{SO}(n) \subset \mathrm{U}(n)$. Then, taking g and z as in (4.29),

$$\Phi_\infty(\gamma g, s) = \det(v)^{s+\rho_n} \cdot |\det(cz+d)|^{-s-\rho_n} \cdot \phi(\mathbf{k}), \tag{4.38}$$

where $\mathbf{k} = \mathbf{k}(\gamma, z)$ is as in (4.30). To find $\mathbf{k}(\gamma, z)$ more explicitly, observe that if we set

$$X = i\,{}^t\alpha^{-1}\mathbf{k} = i \cdot (c\bar{z}+d)\,{}^tv^{-1}, \tag{4.39}$$

as in (4.31), then

$$\bar{X}^{-1} = i\,{}^t\mathbf{k}\,{}^t\alpha. \tag{4.40}$$

Thus

$$\begin{aligned} \bar{X}^{-1}X &= -{}^t\mathbf{k}\cdot\mathbf{k} \qquad (4.41)\\ &= -{}^tv(cz+d)^{-1}(c\bar{z}+d)\,{}^tv^{-1}, \qquad (4.42) \end{aligned}$$

and

$$ {}^t\mathbf{k}\mathbf{k} = {}^tv(cz+d)^{-1}(c\bar{z}+d)\,{}^tv^{-1}. \tag{4.43}$$

Note that, by left $\mathrm{SO}(n)$ invariance, the function ϕ depends only on ${}^t\mathbf{k}\mathbf{k}$ and on $\det(\mathbf{k})$. Thus, in general,

$$E(g, s, \Phi) = \det(y)^{\frac{1}{2}(s+\rho_n)} \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} |\det(cz+d)|^{-s-\rho_n}\phi_\infty(\mathbf{k}(\gamma, z)) \prod_{p|N}\phi_p(\gamma). \tag{4.44}$$

We will not continue our discussion of the general case.

Suppose, now, that $N = 1$. Note that the character χ must be everywhere unramified and hence trivial, since $F = \mathbb{Q}$. Then, for g and z related by (4.29), for $\mathbb{P}_\infty(s) = \mathbb{P}^\ell_\infty(s)$, and for $\Gamma_\infty = \Gamma\cap P(\mathbb{Q})$, we have

$$E(g, s, \mathbb{P}) = \det(y)^\ell \sum_{\Gamma_\infty\backslash\Gamma} \det(cz+d)^{-\ell}|\det(cz+d)|^{-s-\rho_n+\ell}\det(y)^{\frac{1}{2}(s+\rho_n-\ell)} \tag{4.45}$$

$$= \det(y)^\ell E_{\mathrm{Classical}}(z, s+\rho_n-\ell). \tag{4.46}$$

Here

$$E_{\mathrm{Classical}}(z, s) = \sum_{(c,d)} \det(cz+d)^{-\ell}|\det(cz+d)|^{-s}\ \det(y)^{\frac{1}{2}s}, \tag{4.47}$$

is the most classical Siegel Eisenstein series of weight ℓ.

It is amusing to determine when this level 1 (i.e., $N = 1$) Eisenstein series is *incoherent*, say at the point $s = 0$. We assume that $n > 1$ is odd and set $m = n+1 = 2r$. Recall that $\chi = 1$. Also recall that, in the non-archimedean case, the local (unitarizable) induced representations decompose as:

$$I_{n,p}(0) = R_{n,p}(V_{r,r}) \oplus R_{n,p}(V_B), \tag{4.48}$$

where $V_{r,r}$ is the split space and V_B is the ‘quaternionic, space (i.e., the direct sum of the norm form from the quaternion algebra $B = B_p$ over $\mathbb{Q}_p$ and $V_{r-2,r-2}$. In the archimedean case:

$$I_{n,\infty}(0) = R_n(m,0) \oplus R_n(m-2,2) \oplus \cdots \oplus R_n(r,r) \oplus \cdots \oplus R_n(0,m), \quad (4.49)$$

if $m \equiv 0 \mod 4$, and

$$I_{n,\infty}(0) = R_n(m-1,1) \oplus R_n(m-3,3) \oplus \cdots \oplus R_n(r,r) \oplus \cdots \oplus R_n(1,m-1), \quad (4.50)$$

if $m \equiv 2 \mod 4$.

Note that $R_{n,p}(V_{r,r})$ contains the spherical vector $\Phi_p^0(0)$, so that our local components lie in this subspace for every finite place. Since the Hasse invariant of $V_{r,r}$ is

$$\epsilon_p(V_{r,r}) = (-1,-1)_p^{\frac{r(r-1)}{2}}, \quad (4.51)$$

we see that our global section will be *incoherent* precisely when $\Phi_\infty^\ell(0)$ lies in some $R_n(p,q)$ for which

$$\epsilon_\infty(V_{p,q}) = (-1)^{\frac{q(q-1)}{2}} = -\epsilon_\infty(V_{r,r}) = -(-1)^{\frac{r(r-1)}{2}}. \quad (4.52)$$

Note that the condition $\chi_\infty = 1$ implies that ℓ is even. In the range

$$-r \leq \ell \leq r \qquad \ell \text{ even}, \quad (4.53)$$

the vector

$$\Phi_\infty^\ell(0) \in R_n(r+\ell, r-\ell). \quad (4.54)$$

Thus, in this range, (4.52) becomes

$$(-1)^{\frac{(r-\ell)(r-\ell-1)}{2}} = -(-1)^{\frac{r(r-1)}{2}}, \quad (4.55)$$

i.e., simply $\ell \equiv 2 \mod 4$, since ℓ is even. On the other hand, if $\ell \geq r$ (resp. $\ell \leq -r$), then

$$\Phi_\infty^\ell(0) \in \begin{cases} R_n(m,0) & (\text{ resp. } R_n(0,m)) & \text{if } m \equiv 0 \mod 4 \\ R_n(m-1,1) & (\text{ resp. } R_n(1,m-1)) & \text{if } m \equiv 2 \mod 4. \end{cases} \quad (4.56)$$

Proposition 4.4. *Consider the level* 1 *Eisenstein series* $E(g,s,\Phi^\ell)$ *of (even) weight* ℓ *at* $s=0$.

(i) If $-r \leq \ell \leq r$, *then* $E(g,s,\Phi^\ell)$ *is coherent if* $\ell \equiv 0 \mod 4$ *and incoherent if* $\ell \equiv 2 \mod 4$.
(ii) If $\ell \geq r$, *then* $E(g,s,\Phi^\ell)$ *is coherent if* $r \equiv 0, 1 \mod 4$ *and incoherent if* $r \equiv 2, 3 \mod 4$.
(iii) If $\ell \leq -r$, *then* $E(g,s,\Phi^\ell)$ *is coherent if* $r \equiv 0, 3 \mod 4$ *and incoherent if* $r \equiv 1, 2 \mod 4$.

In particular, assume that $m = n+1 \equiv 0 \mod 4$, and take $\ell = r = \frac{n+1}{2}$, so that $\Phi_\infty^\ell(0) \in R_n(m,0)$. If $n \equiv 7 \mod 8$, then $\ell \equiv 0 \mod 4$ and $E(g,s,\Phi^\ell)$ is coherent at $s=0$. Its value $E(g,0,\Phi^\ell)$ is a non-zero holomorphic Siegel

Eisenstein series which is expressible as the usual average over classes in a genus of positive definite quadratic forms of dimension $m = n+1$ and character $\chi_V = 1$. On the other hand, if $n \equiv 3 \mod 8$, then $\ell \equiv 2 \mod 4$ and $E(g,s,\Phi^\ell)$ is incoherent at $s=0$. The explicit calculation of $E'(g,0,\Phi^\ell)$ is then of interest! The case $n=3$, and its generalizations to arbitrary N play a key role in [10], and [13] and the generalization of [11] mentioned in Section 12 of [10].

As another example of an incoherent Eisenstein series, suppose that $n=2$, $N=1$ (so $\chi=1$), and $\ell=2$, so that we are considering the level 1 Eisenstein series of weight 2 and genus 2. In this case, we consider the point $s_0 = 2-\frac{3}{2} = \frac{1}{2}$. Then,

$$I_{2,p}(\frac{1}{2}) = R_{2,p}(V_{2,2}) + R_{2,p}(V_B), \tag{4.57}$$

where the sum is no longer direct. However $\Phi_p^0(\frac{1}{2})$ lies in $R_{2,p}(V_{2,2})$ and *not* in $R_{2,p}(V_B)$. The product of the non-archimedean Hasse invariants of the spaces $V_{2,2}$ must equal the archimedean Hasse invariant $\epsilon_\infty(V_{2,2}) = (-1,-1)_{\mathbb{R}} = -1$. On the other hand,

$$I_{2,\infty}(\frac{1}{2}) = R_{2,\infty}(2,2) \supsetneq R_{2,\infty}(4,0) + R_{2,\infty}(0,4), \tag{4.58}$$

and the vector $\Phi_\infty^2(\frac{1}{2})$ lies in the submodule $R_{2,\infty}(4,0)$. The global section is not literally incoherent in the sense of Section 2.1, since it lies in the space $\Pi_2(V_{2,2})$. However, the complementary space V_0 to $V_{2,2}$ is just the split binary space $V_{1,1}$, and our section $\Phi^2(\frac{1}{2})$ lies in the kernel of the map $\Pi_2(V_{2,2}) \longrightarrow \Pi_2(V_{1,1})$ of (3.8). Thus, by (ii) of Theorem 3.3 of Section 3.1, $E(g,s,\Phi^2)$ is holomorphic at $s=\frac{1}{2}$. The same remarks apply if we take any $\Phi_\infty(\frac{1}{2}) \in R_{2,\infty}(4,0)$, so that there is a map

$$\Pi_2(\mathcal{C}) = R_{2,\infty}(4,0) \otimes (\otimes_p R_{2,p}(V_{2,2})) \longrightarrow \mathcal{A}(G) \tag{4.59}$$

$$\Phi \mapsto E(g,\frac{1}{2},\Phi). \tag{4.60}$$

This map is only intertwining modulo the image of $\bar{A}_{-1} : \Pi_2(V_{1,1}) \longrightarrow \mathcal{A}(G)$, and thus we obtain an *extension*

$$0 \longrightarrow \Pi_2(V_{1,1}) \longrightarrow Y \longrightarrow \Pi_2(\mathcal{C}) \longrightarrow 0, \tag{4.61}$$

where Y is the inverse image in $\mathcal{A}(G)$ of the image of $\Pi_2(\mathcal{C})$ in $\mathcal{A}(G)/Im(\bar{A}_{-1})$. Note that the vector $\Phi_\infty^2(\frac{1}{2}) \in R_{2,\infty}(4,0)$ is 'holomorphic', i.e., is annihilated by the $\bar{\partial}$ operator:

$$\bar{\partial} f = \sum_i X_i f \cdot \omega_i \tag{4.62}$$

where the sum is over a basis X_i for $\mathfrak{p}_- \subset \mathfrak{g}$, the antiholomorphic tangent space to $\mathfrak{H}_2$ at $i\cdot 1_2$, and ω_i runs over the dual basis for $\mathfrak{p}_-^*$. Since $E(g,\frac{1}{2},\Phi^2) \in Y$, we conclude that

$$\bar{\partial} E(g,\frac{1}{2},\Phi^2) \in \Pi_2(V_{1,1}) \otimes \mathfrak{p}_-^*. \tag{4.63}$$

In fact, the function $E(g, \frac{1}{2}, \Phi^2)$ has recently been written out explicitly as a Fourier series by Kohnen [17] and also by Nagaoka [30]. It would be of interest to give an explicit expression for the binary 'theta integral' (cf. Section 3.1 of [25]) $\bar{\partial} E(g, \frac{1}{2}, \Phi^2)$.

This last example suggests that the definition of incoherent sections in Section 2.1 ought to be slightly extended in the range $n+1 < m \leq 2n$.

Similar non-trivial extensions

$$0 \longrightarrow \mathrm{Im}\,(A_0) \longrightarrow Y \longrightarrow \Pi_n(\mathcal{C}) \longrightarrow 0 \tag{4.64}$$

can be constructed by considering the map

$$\begin{array}{ccc} E'(0) : \Pi_n(\mathcal{C}) & \longrightarrow & \mathcal{A}(G) \\ \| & & \downarrow \\ \overline{E'(0)} : \Pi_n(\mathcal{C}) & \hookrightarrow & \mathcal{A}(G)/\mathrm{Im}(A_0) \end{array} \tag{4.65}$$

where A_0 is as in (3.11), and Y is the inverse image in $\mathcal{A}(G)$ of the image of $\overline{E'(0)}$. The non-triviality follows from the non-vanishing of $\overline{E'(0)}$ together with (ii) of Theorem 2.10 in Section 2.2. Perhaps such extensions have some motivic meaning?

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A Remark on Eisenstein Series

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Summary. We prove meromorphic continuation of Eisenstein series for smooth but not necessarily K-finite sections.

1 Introduction

The theory of Eisenstein series is fundamental for the spectral theory of automorphic forms. It was first developed by Selberg, and was completed by Langlands ([7]; see also [9]). There are several known proofs for the meromorphic continuation of Eisenstein series (apart from very special cases of Eisenstein series which can be expressed in terms of Tate integrals). In all of these proofs it is convenient, if not essential, to assume (in the number field case) that the inducing section is K-finite, to ensure finite dimensionality. However, the analytic properties of Eisenstein series are closely tied to, and at any rate controlled by, those of the intertwining operators. The latter decompose into local intertwining operators. In the archimedean case, a lot is known about the local intertwining operators and no K-finiteness assumption on the section is necessary. It is therefore reasonable to expect that the analytic properties of Eisenstein series for a general smooth section follow from that of K-finite sections. The modest goal of this short note is to carry this out (using the meromorphic continuation of K-finite Eisenstein series as a black box). In fact, by the automatic continuity theorem of Casselman and Wallach ([14, Ch. 11]), at each regular point the Eisenstein series can be extended to smooth sections. This by itself does not suffice to prove meromorphic continuation unless one knows some local uniformity (in the spectral parameter) for the modulus of continuity of the Eisenstein series as a map from the induced representation to the space of automorphic forms. The point is that such uniformity, at least for cuspidal Eisenstein series, is provided by the Maass–Selberg relations together with the properties of the intertwining operators at the archimedean places. As for Eisenstein series induced from other discrete spectrum, their properties can be deduced from those of cuspidal Eisenstein series by Langlands' general theory. We remark however, that there is still an important difference between the analytic properties of K-finite Eisenstein series and smooth ones. The former are meromorphic functions of finite order ([10]), while the latter need not be, already for SL_2. Indeed, even the matrix coefficients of the archimedean intertwining operators (which appear in the constant terms of Eisenstein series) are not necessarily of finite order (cf. [6, Remark 1, p. 625]).

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1.1 Notation

Let G be a reductive group over a number field F. We will often denote $G(F)$ by G as well. (Similarly for other groups.) For simplicity of notation we will assume that G is semisimple. This assumption can be easily lifted. Fix a minimal parabolic subgroup P_0 of G with a Levi decomposition $P_0 = M_0U_0$. Let $P = MU$ be a parabolic of G defined over F and containing P_0 such that $M \supset M_0$. Let T_M be the maximal split torus of the center of M. Thus $T_M \simeq \mathbb{G}_m^r$ where r is the co-rank of P and we let A_M be the subgroup $\mathbb{R}_+$ embedded in $\mathbb{I}_F \simeq T_M(\mathbb{A})$ through the embedding $\mathbb{R} \hookrightarrow \mathbb{A}_{\mathbb{Q}} \hookrightarrow \mathbb{A}_F$. Set $\mathfrak{a}_M^* = X^*(M) \otimes \mathbb{R}$ where $X^*(\cdot)$ denotes the lattice of characters defined over F. This vector space contains the set Σ_P of reduced roots of T_M on the unipotent radical of P. The vector space dual to $\mathfrak{a}_M^*$ will be denoted by $\mathfrak{a}_M$.

Let δ_P be the modulus function of $P(\mathbb{A})$. Finally, choose a maximal compact subgroup $\mathbf{K} = \mathbf{K}_\infty \mathbf{K}_f$ of $G(\mathbb{A})$ which is in a "good position" with respect to M_0 (cf. [9]). In particular, $G(\mathbb{A}) = M(\mathbb{A})U(\mathbb{A})\mathbf{K}$ and $M(\mathbb{A}) \cap \mathbf{K}$ is a maximal compact of $M(\mathbb{A})$. We let $H = H_M : G(\mathbb{A}) \to \mathfrak{a}_M$ be the left-$U(\mathbb{A})$, right-$\mathbf{K}$ invariant function on $G(\mathbb{A})$ so that

$$e^{\langle \chi, H(m) \rangle} = \prod |\chi(m_v)|_v$$

for all $m = (m_v)_v \in M(\mathbb{A})$ and $\chi \in X^*(M)$.

Let $\mathfrak{S}$ be a locally finite set of affine hyperplanes of $\mathfrak{a}_{M,\mathbb{C}}^*$ whose vector part is defined over $\mathbb{R}$. Let $\mathcal{P}_{\mathfrak{S}} = \mathcal{P}_{\mathfrak{S}}(\mathfrak{a}_{M,\mathbb{C}}^*)$ be the set of non-zero polynomials on $\mathfrak{a}_{M,\mathbb{C}}^*$ obtained as products of linear functions, each vanishing on a hyperplane in $\mathfrak{S}$. We denote by $\mathcal{M}_{\mathfrak{S}} = \mathcal{M}_{\mathfrak{S}}(\mathfrak{a}_{M,\mathbb{C}}^*)$ the space of meromorphic functions on $\mathfrak{a}_{M,\mathbb{C}}^*$ with polynomial singularities in $\mathfrak{S}$ ([9, V.1.2]). Thus $f \in \mathcal{M}_{\mathfrak{S}}$ if for any $\lambda \in \mathfrak{a}_{M,\mathbb{C}}^*$ there exist $R \in \mathcal{P}_{\mathfrak{S}}$ and a neighborhood of λ on which Rf is holomorphic (or strictly speaking, coincides with a holomorphic function in the complement of $\cup\mathfrak{S}$).

Let $\mathcal{F}$ be the union of an increasing sequence of Fréchet spaces $\mathcal{F}_n$ embedded continuously in one another, with the inductive limit topology. We consider $\mathcal{F}_n$ as a (not necessarily closed) subspace of $\mathcal{F}$. We will always assume in addition that $\mathcal{F}$ is Hausdorff. (This is the case if $\mathcal{F}$ is the strict inductive limit of the $\mathcal{F}_n$'s, but not in general.) In this case, any continuous linear map $f : V \to \mathcal{F}$ from a Fréchet space V factors through a continuous linear map from V to $\mathcal{F}_n$ for some n ([5, I, §3.3, Proposition 1]). Consider the space $\mathcal{M}_{\mathfrak{S}}(\mathcal{F}) = \mathcal{M}_{\mathfrak{S}}(\mathfrak{a}^*_{M,\mathbb{C}}; \mathcal{F})$ of meromorphic functions from $\mathfrak{a}^*_{M,\mathbb{C}}$ to $\mathcal{F}$ with polynomial singularities in $\mathfrak{S}$, which is defined as follows. For every bounded open set $U \subset \mathfrak{a}^*_{M,\mathbb{C}}$, fix $R \in \mathcal{P}_{\mathfrak{S}}$ which vanishes on the finitely many hyperplanes in $\mathfrak{S}$ that intersect U, and let $\mathcal{M}_{\mathfrak{S}}(U; \mathcal{F})$ be the increasing union of the spaces

$$\{f : U \mapsto \mathcal{F}_n | R^n f \text{ is holomorphic}\}$$

with the inductive limit topology, where on each such space we take the semi-norms

$$\sup_{\lambda \in U} |R^n(\lambda)\mu(f(\lambda))| .$$

Here μ is a semi-norm of $\mathcal{F}_n$. Note that $\mathcal{M}_{\mathfrak{S}}(U; \mathcal{F})$ is Hausdorff, because for any $x \in U \setminus \cup\mathfrak{S}$ the map $x \mapsto f(x)$ is a continuous map into a Hausdorff space, and these maps separate the points in $\mathcal{M}_{\mathfrak{S}}(U; \mathcal{F})$. By definition, $\mathcal{M}_{\mathfrak{S}}(\mathfrak{a}^*_{M,\mathbb{C}}; \mathcal{F})$ is the space of $\mathcal{F}$-valued functions on $\mathfrak{a}^*_{M,\mathbb{C}}$ whose restriction to any such U lies in $\mathcal{M}_{\mathfrak{S}}(U; \mathcal{F})$. It is equipped with the coarsest topology for which the restriction maps $\mathcal{M}_{\mathfrak{S}}(\mathcal{F}) \to \mathcal{M}_{\mathfrak{S}}(U; \mathcal{F})$ are continuous. Note that Cauchy's theorem and integral formula apply to any holomorphic function in $\mathcal{M}_{\mathfrak{S}}(\mathcal{F})$, because they hold for holomorphic functions with values in a Fréchet space — cf. [11].

Let $A_{\mathrm{mod}}(G\backslash G(\mathbb{A}))$ be the space of smooth functions on $G\backslash G(\mathbb{A})$ which are of uniform moderate growth ([9, I.2.3]). It is the inductive limit of the Fréchet spaces $A_{\mathrm{mod}}(G\backslash G(\mathbb{A}))_n$ defined by the semi-norms

$$\|f\|_{n,X} = \sup_{g \in \mathfrak{s}} |\delta(X)f(g)| \, \|g\|^{-n}$$

for any $X \in U(\mathfrak{g}_\infty)$, where $\mathfrak{s}$ is a Siegel set for $G\backslash G(\mathbb{A})$ and δ denotes the action of the universal enveloping algebra $U(\mathfrak{g}_\infty)$ of the Lie algebra $\mathfrak{g}_\infty$ of $G(F_\infty)$ on $A_{\mathrm{mod}}(G\backslash G(\mathbb{A}))$. The space $A_{\mathrm{mod}}(G\backslash G(\mathbb{A}))$ is Hausdorff because any point evaluation is a continuous linear form.

2 Eisenstein series and intertwining operators

Let V be an irreducible subspace of $L^2(A_M M(F)\backslash M(\mathbb{A}))$ and let (π, V) be the corresponding representation of $M(\mathbb{A})$. Thus π is an automorphic representation of $M(\mathbb{A})$ whose central character is trivial on A_M. Let $\mathcal{A}^\pi_P$ denote the space of automorphic forms φ on $U(\mathbb{A})M\backslash G(\mathbb{A})$ such that for all $k \in \mathbf{K}$

the function $m \mapsto \delta_P(m)^{-\frac{1}{2}}\varphi(mk)$ belongs to the space of π. (This differs from the perhaps more common usage of $\mathcal{A}_P^\pi$ where $m \mapsto \delta_P(m)^{-\frac{1}{2}}\varphi(mk)$ is only required to belong to the π-isotypic part of $L^2(A_M M(F)\backslash M(\mathbb{A}))$.) The automorphic realization of π gives rise to an identification of $\mathcal{A}_P^\pi$ with the **K**-finite part of the induced space $I(\pi) = \operatorname{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})}\pi$. Set $\varphi_\lambda(g) = \varphi(g)e^{\langle\lambda,H(g)\rangle}$ for any $\varphi \in \mathcal{A}_P^\pi$, $\lambda \in \mathfrak{a}_{M,\mathbb{C}}^*$. The map $\varphi \mapsto \varphi_\lambda$ identifies $I(\pi)$ (as a **K**-module) with any $I(\pi,\lambda) = I_P(\pi,\lambda) = \operatorname{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})}\pi \cdot e^{\langle\lambda,H(\cdot)\rangle}$.

For any $\varphi \in \mathcal{A}_P^\pi$ consider the Eisenstein series which is the meromorphic continuation of the series

$$E(g,\varphi,\lambda) = E_P(g,\varphi,\lambda) = \sum_{\gamma\in P\backslash G}\varphi_\lambda(\gamma g)$$

which converges for $\Re(\lambda)$ sufficiently regular in the positive Weyl chamber of $\mathfrak{a}_M^*$. Whenever regular, $\varphi \mapsto E(\varphi,\lambda)$ defines an intertwining map from $I(\pi,\lambda)_{\mathbf{K}-\mathrm{fin}}$ into the space of automorphic forms on $G(\mathbb{A})$.

Denote by $\|\cdot\|_\pi$ the Hilbert norm on the space of π. Exactly as in the archimedean situation (cf. [14, 10.1.1]) the space $I(\pi)^\infty$ is a Fréchet space with respect to the semi-norms

$$\|X\varphi\|_\infty$$

where

$$\|\varphi\|_\infty = \max_{k\in\mathbf{K}}\|\varphi(k)\|_\pi$$

and X ranges over the universal enveloping algebra of the complexified Lie algebra $\mathfrak{k}_\mathbb{C}$ of K_∞. Moreover, the argument of [loc. cit] immediately shows the following.

Lemma 2.1. *For any $X \in U(\mathfrak{g}_\mathbb{C})$ there exist $n \in \mathbb{N}$ and a continuous semi-norm μ such that*

$$\|I(X,\pi,\lambda)\varphi\|_\infty \le (1+\|\lambda\|)^n\mu(\varphi)$$

for any $\varphi \in I(\pi)^\infty$ and $\lambda \in \mathfrak{a}_{M,\mathbb{C}}^*$. Here $I(X,\pi,\lambda)$ denotes the action of $U(\mathfrak{g}_\mathbb{C})$ on $I(\pi,\lambda)^\infty$ where $\mathfrak{g}_\mathbb{C}$ is the complexification of the Lie algebra of $G(\mathbb{R})$.

Fix an open subgroup K_0 of $\mathbf{K}_f$ and denote the K_0-part of a representation V by V^{K_0}. Our goal is to prove the following result.

Theorem 2.2. *There exists a locally finite collection of hyperplanes $\mathfrak{S} = \mathfrak{S}_{K_0}$ such that the map $\varphi \mapsto E_P(\varphi,\lambda)$ extends to a continuous linear map (still denoted by $E(\varphi,\lambda)$) from $I(\pi)^{\infty,K_0}$ to $\mathcal{M}_\mathfrak{S}(A_{\mathrm{mod}}(G\backslash G(\mathbb{A})))$.*

Explicitly, this means that for any compact set Λ of $\mathfrak{a}_{M,C}^*$ there exist $R \in \mathcal{P}_\mathfrak{S}$, $n \in \mathbb{N}$, and for each $X \in U(\mathfrak{g}_\infty)$ a semi-norm μ such that

$$\|R(\lambda)\delta(X)E(\varphi,\lambda)\|_n \le \mu(\varphi) \tag{2.1}$$

for all $\varphi \in (\mathcal{A}_P^\pi)^{K_0}$ and $\lambda \in \Lambda$.

Theorem 2.2 will be proved in this section for the case where V is cuspidal, and for the general case in the next section. Before proving Theorem 2.2 let us mention a corollary thereof. Let $A_{\mathrm{dec}}(G\backslash G(\mathbb{A}))$ be the Fréchet space of rapidly decreasing functions on $G\backslash G(\mathbb{A})$ with the semi-norms $\sup_{g\in\mathfrak{s}} f(g)\|g\|^n$, $n = 1, 2, \ldots$ Arthur's truncation operator Λ^T defines a continuous linear map from $A_{\mathrm{mod}}(G\backslash G(\mathbb{A}))$ to $A_{\mathrm{dec}}(G\backslash G(\mathbb{A}))$ ([1, Lemma 1.4]). We conclude

Corollary 2.3. *The map* $\varphi \mapsto \Lambda^T E(\varphi, \lambda)$ *is a continuous linear map from* $I(\pi)^{\infty,K_0}$ *to* $\mathcal{M}_{\mathfrak{S}}(A_{\mathrm{dec}}(G\backslash G(\mathbb{A})))$.

We recall the intertwining operators $M_{Q|P}(\pi, \lambda)$ as defined for example in [2]. Here Q belongs to the set $\mathcal{P}(M)$ of the finitely many parabolic subgroups which contain M as a Levi subgroup.

These intertwining operators admit local analogues. In the p-adic case the matrix coefficients of the local intertwining operators are rational functions of $q^{\langle\lambda,\alpha^\vee\rangle}$, $\alpha \in \Sigma_P$, where q is the cardinality of the residue field ([12] and [13]). Consider now the archimedean case. In the ensuing discussion the notation and the objects will pertain to real reductive groups. Recall ([14, Theorem 10.1.5]) that for some ρ in the positive Weyl chamber there exist a non-zero scalar-valued polynomial $b(\lambda)$ and a polynomial $D(\lambda)$ with values in (a finite dimensional subspace of) $U(\mathfrak{g}_\mathbb{C})$ such that

$$b(\lambda)M_{Q|P}(\pi, \lambda) = M_{Q|P}(\pi, \lambda + \rho)I(D(\lambda), \pi, \lambda + \rho). \tag{2.2}$$

The functions b and D depend on Q (as well as on π of course). However, we can choose b of the form

$$b(\lambda) = \prod_{\alpha\in\Sigma_P\cap\Sigma_{\bar{Q}}} b_\alpha(\langle\lambda, \alpha^\vee\rangle)$$

for some polynomial functions b_α, where $\bar{Q}$ is parabolic subgroup in $\mathcal{P}(M)$ opposite to Q. For $\Re(\lambda)$ sufficiently regular in the positive Weyl chamber, $M_{Q|P}(\pi, \lambda)$ is absolutely convergent and in fact, $M_{Q|P}(\pi, \lambda)$ is a bounded operator with respect to $\|\cdot\|_\infty$, independently of λ ([ibid., Lemma 10.1.11]). We immediately infer:

Corollary 2.4. *Let* $\mathfrak{S}$ *consist of the hyperplanes* $\langle\lambda + k\rho, \alpha^\vee\rangle = c$ *where* $k \in \mathbb{N}$ *and* c *is a root of* b_α. *Then all matrix coefficients of* $M_{Q|P}(\pi, \lambda)$ *lie in* $\mathcal{M}_\mathfrak{S}$. *Moreover, for* λ *in a compact set, there exist a semi-norm* μ *and* $R \in \mathcal{P}_\mathfrak{S}$ *such that*

$$\|R(\lambda)M_{Q|P}(\pi, \lambda)\varphi\|_\infty \le \mu(\varphi) \tag{2.3}$$

for all $\varphi \in I(\pi)^\infty$.

In fact, R and μ can be chosen uniformly on any cone $\langle\lambda, \alpha^\vee\rangle > d$, $\forall\alpha \in \Delta_P$.

Back to the global setup, we can write, for each $Q \in \mathcal{P}(M)$, the restriction of $M_{Q|P}(\pi, \lambda)$ to $I(\pi)^{\infty,K_0}$ as

$$m_{\pi,Q|P}(\lambda) \prod_{v \in S} M_{Q|P,v}(\pi_v, \lambda)$$

where S is a finite set of places, depending on K_0, containing all the archimedean ones. By the theory of Eisenstein series, any K-finite matrix coefficient of $M_{Q|P}(\pi, \lambda)$ lies in $\mathcal{M}_{\mathfrak{S}}$ for a suitable $\mathfrak{S}$. By the properties of the local intertwining operators mentioned above, it follows that $m_{\pi,Q|P}$ belongs to $\mathcal{M}_{\mathfrak{S}}$. In fact, we can even conclude that for a suitable $\mathfrak{S}$, all matrix coefficients of $M_{Q|P}(\pi, \lambda)$ (as an operator on $I(\pi)^{\infty, K_0}$) lie in $\mathcal{M}_{\mathfrak{S}}$, and the relation (2.3) immediately carries over to the global setting (for $\varphi \in I(\pi)^{\infty, K_0}$).

For the rest of this section we assume that the space of π consists of cusp forms, and prove Theorem 2.2 in this generality. The general case will be treated in the next section.

We first note that from [9, Remark IV.4.4] and the properties of the intertwining operators mentioned above, it follows that for any $\varphi \in I(\pi)^{K_0}_{\mathbf{K}-\mathrm{fin}}$, $E(\varphi, \lambda) \in \mathcal{M}_{\mathfrak{S}}(L^2_{loc}(G\backslash G(\mathbb{A})))$. Following the argument of [8, §6] we will prove a more precise and technical statement, bounding the L^2-norm of the truncated Eisenstein series $\Lambda^T E(g, \varphi, \lambda)$ (cf. [1, §4]). This will enable us to deduce Theorem 2.2 in the cuspidal case. Henceforth we assume that λ is confined to a compact set which will be fixed throughout. It is understood that all relevant objects (such as elements $R \in \mathcal{P}_{\mathfrak{S}}$ or semi-norms μ) implicitly depend on this compact set.

Proposition 2.5. *There exist an element $R \in \mathcal{P}_{\mathfrak{S}}$, a semi-norm μ on $I(\pi)$ and $C \geq 1$ such that*

$$\|R(\lambda)\Lambda^T E(\cdot, \varphi, \lambda)\|_{L^2(G\backslash G(\mathbb{A}))} \leq C^{\|T\|}\mu(\varphi) \tag{2.4}$$

for all $\varphi \in I(\pi)^\infty$ and all sufficiently regular T. More generally, for any $X \in U(\mathfrak{g}_{\infty,\mathbb{C}})$ a similar upper bound (with μ depending on X) holds for $\|R(\lambda)\Lambda^T E(\cdot, I(X, \pi, \lambda)\varphi, \lambda)\|_{L^2(G\backslash G(\mathbb{A}))}$.

Proof. First note that the second part follows from the first in view of Lemma 2.1. Set

$$\|\Lambda^T E(\cdot, \varphi, \lambda)\|^2_{L^2(G\backslash G(\mathbb{A}))} = (\Omega^T(\lambda)\varphi, \varphi).$$

By the Maass–Selberg relations, the operator $\Omega^T(\lambda)$ is given by the sum over the representatives $s \in W/W_M$ such that $sMs^{-1} = M$ of the value at $\lambda' = \lambda$ of

$$\sum_{Q \in \mathcal{P}(M_P)} M_{Q|P}(\lambda)^* M_{Q|P}(s\lambda') M_{P|P}(s, \lambda') e^{\langle s\lambda' + \overline{\lambda}, Y_Q(T)\rangle} \theta_Q(s\lambda' + \overline{\lambda})^{-1}$$

where Y_Q is a certain affine function of T (explicitly described in [2, p. 1295-6]) and $\theta_Q(\lambda) = \prod_{\alpha \in \Delta_Q} \langle \lambda, \alpha^\vee \rangle$, with the product taken over the simple roots of T_M in the unipotent radical of Q. Unlike in [loc. cit.], we do not assume that $\lambda \in i\mathfrak{a}^*_M$. Recall also ([ibid., p. 1310]) the (G, M)-families (in Λ)

$$[rl]\ \mathcal{M}_Q(\lambda,\Lambda) = M_{Q|P}(\lambda)^* M_{Q|P}(-\overline{\lambda}+\Lambda)$$
$$c_Q(T,\Lambda) \ = e^{\langle \Lambda, Y_Q(T)\rangle}$$
$$\mathcal{M}_Q^T(\lambda,\Lambda) = c_Q(T,\Lambda)\mathcal{M}_Q(\lambda,\Lambda).$$

Then $\Omega^T(\lambda)$ is the sum over s of the value at $\lambda' = \lambda$ of

$$\sum_{Q\in\mathcal{P}(M_P)} \mathcal{M}_Q(\lambda, s\lambda'+\overline{\lambda})c_Q(T,s\lambda'+\overline{\lambda})M_{P|P}(s,\lambda')\theta_Q(s\lambda'+\overline{\lambda})^{-1}$$

which in the notation of [ibid.] is $\mathcal{M}_M^T(\lambda, s\lambda+\overline{\lambda})M_{P|P}(s,\lambda)$. The global analogue of Corollary 2.4 applies to $M_{P|P}(s,\lambda)$ (on $I(\pi)^{\infty,K_0}$). It remains to show that for λ, Λ in a compact set there exist $R\in\mathcal{P}_{\mathfrak{S}}$, $C\geq 1$, and for any X a semi-norm μ such that

$$\|R(\lambda)R(-\overline{\lambda}+\Lambda)\mathcal{M}_M^T(\lambda,\Lambda)I(X,\pi,\lambda)\varphi\|_\infty \leq \mu(\varphi)C^{\|T\|} \tag{2.5}$$

for all φ. Fix a vector $\xi\in\mathfrak{a}_M^*$ such that $\langle\xi,\alpha^\vee\rangle\neq 0$ for all $\alpha\in\Sigma_P$. Write $R(\lambda)R(-\overline{\lambda}+\Lambda)\mathcal{M}_M^T(\lambda,\Lambda)$ as a Cauchy integral

$$\frac{1}{2\pi \mathrm{i}}\oint \frac{R(\lambda)R(-\overline{\lambda}+\Lambda+z\xi)\mathcal{M}_M^T(\lambda,\Lambda+z\xi)}{z}\ dz$$

over a circle C_r of radius r centered at 0 (for an appropriate $R\in\mathcal{P}_{\mathfrak{S}}$). If r is sufficiently large with respect to λ and Λ, the denominators $\theta_Q(\Lambda+z\xi)$ appearing in the expression for the integrand are bounded away from zero. The dependence on T is controlled by c_Q. To conclude (2.5) it remains to apply the global analogue of (2.3). □

Fix a compact set $B\subset G(\mathbb{A})$ which is left and right **K**-invariant. We first prove that there exist $n\in\mathbb{N}$, $R\in\mathcal{P}_{\mathfrak{S}}$ and a semi-norm μ such that

$$\|R(\lambda)E(I(f,\pi,\lambda)\varphi,\lambda)\|_n \leq \|f\|_\infty\mu(\varphi) \tag{2.6}$$

for all bi-K-finite f supported in B and K-finite φ.[1] Indeed, for T regular enough (depending only on B and g) $E(g,I(f,\lambda)\varphi,\lambda)$ equals

$$\int_{G(\mathbb{A})} f(g^{-1}x)E(x,\varphi,\lambda)\ dx = \int_{G(\mathbb{A})} f(g^{-1}x)\Lambda^T E(x,\varphi,\lambda)\ dx.$$

In fact, we can choose T so that $\|T\|$ is bounded by a constant multiple of $1+\log\|g\|$ (cf. [8, Lemma 6.2]). We rewrite the above as

[1] Once Theorem 2.2 is established, (2.6) and (2.1) hold without the K-finiteness restriction on f and φ, by continuity.

$$\int_{G\backslash G(\mathbb{A})} \left(\sum_{\gamma \in G} f(g^{-1}\gamma x) \right) \Lambda^T E(x, \varphi, \lambda) \, dx.$$

By [9, I.2.4], for some r and c (depending on B) we have

$$\left| \sum_{\gamma \in G} f(g^{-1}\gamma x) \right| \leq c \|g\|^r \|f\|_\infty$$

for all $x, g \in G(\mathbb{A})$. Hence, by Cauchy–Schwartz

$$|E(g, I(f, \pi, \lambda)\varphi, \lambda)| \leq c' \|f\|_\infty \|g\|^r \|\Lambda^T E(\cdot, \varphi, \lambda)\|_2.$$

We conclude (2.6) from Proposition 2.5.

More generally, for any $X \in U(\mathfrak{g}_{\infty,\mathbb{C}})$ we have a similar bound for $E(I(f, \pi, \lambda)I(X, \pi, \lambda)\varphi, \lambda)$.

Thus, the map $f \to E(I(f, \pi, \lambda)\varphi, \lambda)$ extends to $C_c(G(\mathbb{A}))$.

By [4, §4] there exist $f_1 \in C_c^\infty(G(F_\infty))$, $f_2 \in C_c(G(F_\infty))$ and $Z \in U(\mathfrak{g}_{\infty,\mathbb{C}})$ such that $f_1 + f_2 \star Z$ is equal to the Dirac distribution at the identity. (In fact, we can choose $f_2 \in C_c^m(G(F_\infty))$ for any given m.) Hence, if $F_i = \mathrm{vol}(K_0)^{-1} \cdot f_i \otimes 1_{K_0}$, $i = 1, 2$ where 1_{K_0} is the characteristic function of K_0 then we have

$$E(\varphi, \lambda) = E(I(F_1, \lambda)\varphi, \lambda) + E(I(F_2, \lambda)I(Z, \lambda)\varphi, \lambda).$$

It follows that there exist $R \in \mathcal{P}_{\mathfrak{S}}$, $n \in \mathbb{N}$ and a semi-norm μ such that

$$\|R(\lambda)E(\varphi, \lambda)\|_n \leq \mu(\varphi).$$

A similar estimate (with μ depending on X) holds for $\delta(X)E(\varphi, \lambda)$. This concludes the proof of Theorem 2.2.

Remark 2.6. Note that in the case $P = G$ the above argument shows that the map from the smooth part of π to $A_{\mathrm{mod}}(G\backslash G(\mathbb{A}))$ (and therefore to $A_{\mathrm{dec}}(G\backslash G(\mathbb{A}))$) is continuous. Of course, this also follows from the automatic continuity theorem (which we never used).

3 Non-cuspidal Eisenstein series

It remains to prove Theorem 2.2 without the cuspidal assumption on π. We will derive it from the cuspidal case using Langlands' theory which expresses $E_P(\varphi, \lambda)$ in terms of residues of cuspidal Eisenstein series. We briefly recall how this is done. (See [9, Ch. VI] for a precise statement.) First, recall the notion of *residue data* defined in [9, V.1.3]. It is a linear map (depending on some choices) $\mathrm{Res}_{V'} : \mathcal{M}_{\mathfrak{S}}(\mathfrak{a}^*_{M,\mathbb{C}}; \mathcal{F}) \to \mathcal{M}_{\mathfrak{S}'}(V'; \mathcal{F})$ where V' is an intersection of hyperplanes in $\mathfrak{S}$ and

$$\mathfrak{S}' = \{\sigma \cap V' : \sigma \in \mathfrak{S}, \sigma \not\supset V'\}.$$

Note that $\mathrm{Res}_{V'}$ is continuous. This amounts to showing that for any small open $U \subset \mathfrak{a}^*_{M,\mathbb{C}}$ and $R \in \mathcal{P}_{\mathfrak{S}}$ there exist a small open $U' \subset U \cap V'$, $R' \in \mathcal{P}_{\mathfrak{S}'}$ and a constant c such that for any semi-norm μ of $\mathcal{F}_n$, $n \in \mathbb{N}$

$$\sup_{U'} \mu(R'(\lambda') \,\mathrm{Res}_{V'} f(\lambda')) \leq c \sup_{U} \mu(R(\lambda) f(\lambda))$$

for any $f \in \mathcal{M}_{\mathfrak{S}}(\mathcal{F})$. Here the restriction of Rf to U is holomorphic with values in $\mathcal{F}_n$. This immediately follows from the discussion of [loc. cit.] and Cauchy's integral formula. It will be useful to consider also the space of functions with polynomial singularities on a given open set of $\mathfrak{a}^*_{M,\mathbb{C}}$. A similar statement holds for taking residues in this setup.

Let B be a parabolic subgroup of G and σ a cuspidal representation of its Levi part. Fix $R \gg 0$ and consider the Fréchet space $\mathcal{PW}^R(\mathfrak{a}^*_{B,\mathbb{C}}; I(\sigma)^\infty)$ consisting of holomorphic functions φ on the tube $\mathfrak{T} = \mathfrak{T}_R = \{\lambda \in \mathfrak{a}^*_{B,\mathbb{C}} : \|\Re\lambda\| < R\}$ with values in $I(\sigma)^\infty$ such that the norms

$$\sup_{\mathfrak{T}} \|I(X, \sigma, \lambda)\varphi(\lambda)\|_\infty (1 + \|\lambda\|)^n, n = 1, 2, \ldots, X \in U(\mathfrak{g}_\mathbb{C})$$

are finite. (It is enough to take $X \in U(\mathfrak{k}_\mathbb{C})$ by Lemma 2.1.) This space has an action of $G(\mathbb{A})$ given by

$$g\varphi(\lambda) = I(g, \sigma, \lambda)\varphi(\lambda).$$

It is easy to see that $\mathcal{PW}(\mathfrak{a}^*_{B,\mathbb{C}}; I(\sigma)^\infty)$ is of *moderate growth* in the sense of [14, 11.5.1]. The argument is similar to that of the Lemma in [loc. cit.] and will be omitted. Using Theorem 2.2 in the cuspidal case we obtain a continuous linear map $\varphi \mapsto E_B(\varphi(\lambda), \lambda)$ from $\mathcal{PW}^R(\mathfrak{a}^*_{B,\mathbb{C}}; I(\sigma)^\infty)$ to $\mathcal{M}_{\mathfrak{S}}(\mathfrak{T}; A_{\mathrm{mod}}(G\backslash G(\mathbb{A})))$. A similar statement holds for the Eisenstein series $E^P_B(\varphi(\Lambda), \Lambda)$, $\Lambda \in (\mathfrak{a}^P_B)^*_\mathbb{C}$ where in its defining series, the sum is taken over $\gamma \in B\backslash P$.

Let χ be the cuspidal data pertaining to π. Then $\mathcal{A}^\pi_P$ is contained in the space $\mathcal{A}^2_{P,\chi}$ of automorphic forms on $U(\mathbb{A})M\backslash G(\mathbb{A})$ having cuspidal data χ such that $\varphi(ag) = \delta_P(a)^{\frac{1}{2}}\varphi(g)$ for all $a \in A_M$ and $\int_{U(\mathbb{A})MA_M\backslash G(\mathbb{A})} |\varphi(g)|^2 \, dg < \infty$. Fix K_0 as before and consider the space

$$\mathcal{PW}^R_\chi = \oplus_{(B,\sigma)\in\chi, B\subset P} \mathcal{PW}^R((\mathfrak{a}^P_B)^*_\mathbb{C}, I(\sigma)^{\infty,K_0})$$

and its subspace $\mathcal{PW}^R_{\chi,K-\mathrm{fin}}$ of $\mathbf{K}$-finite vectors. By [9] (or alternatively, [3, §2]) $\mathcal{A}^{2,K_0}_{P,\chi}$ is the image of the map

$$\mathfrak{e}^P : \mathcal{PW}^R_{\chi,K-\mathrm{fin}} \to A_{\mathrm{mod}}(U(\mathbb{A})M\backslash G(\mathbb{A}))^{K_0}$$

which is given by the sum of residue data of cuspidal Eisenstein series $E^P_B(\varphi(\Lambda), \Lambda)$ ($\Lambda \in (\mathfrak{a}^P_B)^*_\mathbb{C}$) at certain points. This map is also given by a spectral projection applied to the pseudo-Eisenstein series built from φ and it is therefore an intertwining map. Furthermore, by the above, the map extends to a continuous linear map, denoted $\overline{\mathfrak{e}^P}$, on $\mathcal{PW}_\chi^R$. By [14, Theorem 11.8.2] the image of $\overline{\mathfrak{e}^P}$ contains the smooth part of $I(\pi)^{K_0}$, because it contains a dense subspace thereof. By [5, II, §4.7, Proposition 12] we can find a continuous (not necessarily linear) map

$$\iota : I(\pi)^{\infty,K_0} \to \mathcal{PW}_\chi^R$$

such that $\overline{\mathfrak{e}^P} \circ \iota = Id$.

Similarly, we have a continuous linear map

$$\mathfrak{e} : \mathcal{PW}_\chi^R \to \mathcal{M}_{\mathfrak{S}}(\mathfrak{a}_{M,\mathbb{C}}^*; A_{\mathrm{mod}}(G\backslash G(\mathbb{A}))^{K_0}),$$

given by the sum over residue data of the cuspidal Eisenstein series $E_B(\varphi(\Lambda), \lambda + \Lambda)$ $(\Lambda \in (\mathfrak{a}_B^P)_{\mathbb{C}}^*, \lambda \in \mathfrak{a}_{M,\mathbb{C}}^*)$. We have

$$\mathfrak{e}(\varphi, \lambda) = E_P(\mathfrak{e}^P(\varphi), \lambda)$$

on $\mathcal{PW}_{\chi,\mathbf{K}-\mathrm{fin}}^R$. In other words, on $\mathcal{PW}_{\chi,\mathbf{K}-\mathrm{fin}}^R$ $\mathfrak{e}$ factors through $\mathcal{A}_{P,\chi}^{2,K_0}$ as the map E_P. The map $\mathfrak{e}\circ\iota$ is a continuous map from $I(\pi)^{\infty,K_0}$ to $A_{\mathrm{mod}}(G\backslash G(\mathbb{A}))^{K_0}$ which extends E_P. (In particular, it is linear.) This concludes the proof of Theorem 2.2.

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Arithmetic Aspects of the Theta Correspondence and Periods of Modular Forms

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Summary. We review some recent results on the arithmetic of the theta correspondence for certain symplectic-orthogonal dual pairs and some applications to periods and congruences of modular forms. We also propose an integral version of a conjecture on Petersson inner products of modular forms on quaternion algebras over totally real fields.

1 Introduction

The theta correspondence provides a very important method to transfer automorphic forms between different reductive groups. Central to the theory is the important notion of a dual reductive pair. This is a pair of reductive subgroups G and G' contained in an ambient symplectic group H that happen to be the centralizers of each other in H. In such a situation, for every choice of additive character ψ of $\mathbb{A}/\mathbb{Q}$ and for automorphic representations π, π' on G, G' respectively, one may define theta lifts $\Theta(\pi, \psi), \Theta(\pi', \psi)$ that (if nonzero) are automorphic representations on G', G respectively ([13]). In the automorphic theory, it is an important and subtle question to characterize when the lift is non-vanishing. For instance, the non-vanishing could depend on both local conditions (compatibility of ε-factors) and global conditions (non-vanishing of an L-value).

The theta lift has its genesis in the Weil representation of $H(\mathbb{A})$ on a certain Schwartz space $\mathcal{S}(\mathbb{A})$. For any choice of Schartz function $\varphi \in \mathcal{S}(\mathbb{A})$ and vector $f \in \pi$ one may consider the theta lift $\theta(f, \varphi, \psi)$ which is an element of $\Theta(\pi, \psi)$. Now it is often the case that one can define good notions of arithmeticity for elements of π and $\Theta(\pi, \psi)$. Arithmeticity here could mean algebraicity, rationality over a suitable number field or even p-adic integrality. The main problems in the arithmetic theory of the theta correspondence are the following:

Question A: Suppose f is chosen to be arithmetic. For a given canonical choice of φ, is $\theta(f, \varphi, \psi)$ arithmetic (perhaps up to some canonical transcendental period)?

This question has been studied in great detail by Shimura ([23], [24], [25], and [26]), Harris ([5]), and Harris–Kudla ([6] and [7]) in the case of algebraicity and in some cases rationality over suitable number fields. However, the study of such questions in the setting of p-adic integrality is more recent. Before we mention the progress made recently on this subject, we point out that if the answer to question A is affirmative, one may pose the following:

Question B: Suppose that the form f is a p-unit (with respect to some suitable p-adic lattice.) Is $\theta(f, \varphi, \psi)$ a p-unit ? If not, what can be said about the primes p for which $\theta(f, \varphi, \psi)$ has positive p-adic valuation?

Question B is undoubtedly more difficult than Question A and the answer seems to involve certain kinds of congruences of modular forms and μ-invariants of p-adic L-functions. It is also closely related to the classical question of whether certain spaces of modular forms are (integrally) spanned by theta series.

To the authors knowledge, the only known results on Questions A and B in the integral setting are for the following dual pairs.

(i) $(\mathrm{GU}(2), \mathrm{GU}(3))$ ([3])

(ii) $(\mathrm{GL}(2), \mathrm{GO}(B))$ for B a quaternion algebra. (See [18] for the indefinite case with square-free level over $\mathbb{Q}$, work of Emerton [2] for the definite case at prime level over $\mathbb{Q}$, and Hida [11] for the definite case at full level over totally real fields.)

(iii) $(\mathrm{U}(n), \mathrm{U}(n+1))$ ([8] and [9])

(iv) $(\widetilde{\mathrm{SL}(2)}, \mathrm{O}(V))$, for V the space of trace 0 elements in an indefinite quaternion algebra over $\mathbb{Q}$. This case and applications are treated in forthcoming work of the author ([15], [16], and [17].)

In all these cases, there seem to be intimate connections with Iwasawa theory. For instance, (ii) and (iv) use crucially the main conjecture of Iwasawa theory for imaginary quadratic fields, which is a deep theorem of Rubin. The work of Harris, Li and Skinner has as an application the construction of p-adic L-functions for unitary groups and one divisibility of an associated main conjecture. It is certainly to be expected that other cases of the theta correspondence will yield other applications to Iwasawa theory. In addition to this, one also discovers interesting applications to the study of special values of L-functions and integral period relations for modular forms, on which more will be said later.

In this article, we will focus only on cases (ii) and (iv), mainly out of the author's lack of knowledge of the other cases. Here is a brief outline. We begin by describing some questions regarding periods and congruences of modular forms that motivate the study of arithmeticity of the theta correspondence. Next we explain in some detail the integrality of the Jacquet–Langlands correspondence, i.e., the dual pair $(\mathrm{GL}_2, \mathrm{GO}(B))$, in both the indefinite and the definite setting. Some of the results in the definite setting are new in that they do not seem to have appeared elsewhere. This is followed by a brief discussion of integrality results for the Shimura–Shintani–Waldspurger correspondence, i.e. the dual pair $(\widetilde{\mathrm{SL}_2}, PB^\times)$. Here, surprisingly, the results are more complete in the indefinite case. Finally, we propose a conjecture for the Petersson inner products of modular forms on quaternion algebras over totally real fields. Such a conjecture was first made by Shimura up to algebraic factors and mostly proved by Harris in [5]. Ours is a more refined version up to p-adic units that is motivated by Shimura's conjecture and a computation for elliptic curves over $\mathbb{Q}$.

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Note and caution: In order to keep the exposition simple, we will ignore many terms in the formulas that appear below. For instance, we ignore powers of π (3.1415...), other explicit constants, abelian L-functions, etc. Since we will be interested mostly in p-integrality, we use the symbol $\sim$ instead of $=$ to denote equality up to elements that are units at all places above p. The reader may rest assured that every formula that occurs below may be worked out precisely, so that $\sim$ may be replaced by $=$ after throwing in the appropriate constants and terms that we have neglected in the present exposition.

2 Periods of modular forms

Let f be a holomorphic newform of weight $2k$ on $\Gamma_0(N)$ and K_f the field generated by its Hecke eigenvalues. We assume that N is square-free and that we have picked a factorization $N = N^+N^-$ such that N^- is the product of an even number of primes. Let B be the indefinite quaternion algebra over $\mathbb{Q}$ ramified precisely at the primes dividing N^- and g the Jacquet–Langlands lift of f to the Shimura curve X of level N^+ coming from B. We assume that $p \nmid N$ and normalize g (up to a p-adic unit in K_f) using the integral structure provided by sections of the relative dualizing sheaf on the minimal regular model of X at p. Let F be a field containing K_f and if $2k > 2$ we also assume that B splits over F. It is possible then to attach to f and g certain canonical periods $u_\pm(f), u_\pm(g)$ that are well-defined up to p-adic units in F. (See [15] for instance for a definition.) The usual Petersson inner product is related to these periods by

$$\langle f, f\rangle \sim \delta_f \cdot u_+(f) \cdot u_-(f)$$
$$\langle g, g\rangle \sim \delta_g \cdot u_+(g) \cdot u_-(g)$$

for some algebraic numbers δ_f, δ_g. In fact, one expects (and can show under certain hypotheses) that δ_f (resp. δ_g) is a p-integer that "counts" congruences between f and other eigenforms on $X_0(N)$ (resp. between g and other eigenforms on X).

Remark 2.1. In order to make the last statement precise, one needs to define an invariant attached to (f, λ) that measures congruences of f modulo λ. Let R be the ring of integers of a finite extension of $\mathbb{Q}_p$ containing all the Hecke eigenvalues of all forms of level N, set $\mathbb{T}_R = \mathbb{T} \otimes R$ where $\mathbb{T}$ is the usual Hecke algebra over $\mathbb{Z}$ and let $\mathfrak{m}$ be the maximal ideal of $\mathbb{T}_R$ corresponding to the mod λ representation associated to f. If $\mathbb{T}_{\mathfrak{m}}$ denotes the localization of $\mathbb{T}_R$ at $\mathfrak{m}$, $\varphi : \mathbb{T}_{\mathfrak{m}} \to R$ is the eigencharacter of $\mathbb{T}_{\mathfrak{m}}$ corresponding to f and $\wp$ is the kernel of φ, one defines $\eta_f = \varphi(\mathrm{Ann}(\wp))$. One always has $\eta_f \subseteq (\delta_f)$ as ideals in R based upon the theorem of Hida; and under suitable conditions (such as the freeness of certain cohomology groups as $\mathbb{T}_{\mathfrak{m}}$-modules) one has also $\eta_f = (\delta_f)$. (For these results see [10] and the references therein as well as Lemma 4.17 of [1].) The reader may easily convince himself/herself that $l(R/\eta_f)$ is a good measure of congruences satisfied by f. Likewise, one may associate to g an invariant η_g using the ring $\mathbb{T}'_{\mathfrak{m}'}$, where $\mathbb{T}'$ is the Hecke algebra for B and $\mathfrak{m}'$ is the maximal ideal corresponding to (g, λ). Again, one always has $\eta_g \subseteq (\delta_g)$ and under suitable freeness assumptions, one has $\eta_g = (\delta_g)$.

Remark 2.2. The Jacquet–Langlands corespondence implies that $\mathbb{T}'$ is a quotient of $\mathbb{T}$, hence $\eta_f \subseteq \eta_g$ as ideals in R. As a consequence, one sees that under suitable conditions, $\delta_g | \delta_f$ and further, the ratio δ_f / δ_g counts congruences between the Hecke eigencharacter associated to f and other systems of eigenvalues that do not transfer to B via the Jacquet–Langlands correspondence. The example below shows that one may expect $u_\pm(f)/u_\pm(g)$ to be a p-unit if p is not an Eisenstein prime for f. This leads to the expectation that

$$\frac{\langle f, f \rangle}{\langle g, g \rangle} = \frac{\delta_f}{\delta_g}$$

up to Eisenstein primes.

Example 2.3. If $2k = 2$ and $K_f = \mathbb{Q}$, we may pick $F = \mathbb{Q}$. Then f and g correspond to elliptic curves E and E' over $\mathbb{Q}$ that are *strong elliptic curves* for $X_0(N)$ and X respectively, i.e., E, E' are realized as quotients $J_0(N) \to E, Jac(X) \to E'$ with the corresponding dual maps being injective. In this case, $u_\pm(f), u_\pm(g)$ agree with the usual periods of E, E' respectively.

Suppose p is not an Eisenstein prime for f, i.e., the mod p Galois representation associated to E is irreducible. Then, by Faltings' isogeny theorem, one may find an isogeny $E \to E'$ of degree prime to p. It follows that $u_\pm(f) \sim u_\pm(g)$, hence $\frac{\langle f,f \rangle}{\langle g,g \rangle} \sim \frac{\delta_f}{\delta_g}$.

As explained in the preceding remark, the number δ_f/δ_g counts congruences between f and forms that do not transfer to the quaternion algebra

B. In fact, using a result of Ribet and Takahashi [20], one can show more precisely in this case that

$$\frac{\langle f, f\rangle}{\langle g, g\rangle} \sim \prod_{q|N^-} c_q$$

where c_q is the order of the component group of the Neron model of E at q (See [18], Section 2.2.1.) Further one knows that the term c_q counts exactly level-lowering congruences at q, i.e., congruences between f and other forms of level dividing N/q.

Remark 2.4. Our motivation lies in proving such results for forms of arbitrary weight. The difficulty is that one does not know how to geometrically relate the motives associated to modular forms of higher weight and those associated to their quaternionic analogues. As the reader will see, the solution we have in mind to this problem is to use automorphic methods to replace the geometric arguments of the example above.

We now make the following assumptions for the rest of this article:

Assumption I: $p > 2k + 1$.
Assumption II: $p \nmid \tilde{N} := \prod_{q|N} q(q+1)(q-1)$.

It can be shown that Assumption II implies in particular that the Condition (*) below is satisfied by p. (See [18], Lemma 5.1.) That p satisfies this condition is essential in order to apply the integrality criteria (3.5) and Proposition 3.9 below.

Condition (*) There exist infinitely many imaginary quadratic fields K that satisfy any prescribed set of splitting conditions at the primes dividing N, are split at p and have class number prime to p.

Question 2.5. Let p be a prime not dividing N. Does p satisfies Condition (*) even if it does not satisfy Assumption II?

3 Arithmeticity of theta lifts

3.1 The pair $(\mathbf{GL_2}, \mathbf{GO}(B))$

In this section, B is a quaternion algebra over $\mathbb{Q}$ (either definite or indefinite) with discriminant N^- dividing N. We fix isomorphisms $\Phi_q : B \otimes \mathbb{Q}_q \simeq M_2(\mathbb{Q}_q)$ for $q \nmid N^-$. If B is unramified at infinity, we also fix an isomorphism Φ_∞ : $B \otimes \mathbb{R} \simeq M_2(\mathbb{R})$.

In the case when B is ramified at infinity, we first pick a model for B, $B = \mathbb{Q} + \mathbb{Q}a + \mathbb{Q}b + \mathbb{Q}ab$, with $a^2 = -N^-$, $b^2 = -l$ and $ab = -ba$, where l is an auxiliary prime chosen such that

$$\left(\frac{-l}{q}\right) = -1 \text{ if } q \mid N^- \text{ and } q \text{ is odd,}$$
$$l \equiv 3 (\text{mod } 8).$$

Denote by $\mathcal{O}'$ the maximal order in B given by

$$\mathcal{O}' = \mathbb{Z} + \mathbb{Z}\frac{1+b}{2} + \mathbb{Z}\frac{a(1+b)}{2} + \mathbb{Z}\frac{(r+a)b}{l}$$

where r is any integer satisfying $r^2 + N^- \equiv 0 \mod l$. We may assume that the isomorphisms Φ_q are chosen such that $\Phi_q(\mathcal{O}') = M_2(\mathbb{Z}_q)$ for $q \nmid N^-$.

Let $\mathbb{H}$ be the division algebra of Hamilton quaternions, i.e., $\mathbb{H} = \mathbb{R} + \mathbb{R}i + \mathbb{R}j + \mathbb{R}k$ with the relations $i^2 = j^2 = k^2 = -1, ij = -ji = k, jk = -kj = i, ki = -ik = j$ and fix an isomorphism $\Phi_\infty : B \otimes \mathbb{R} \to \mathbb{H}$ characterized by

$$\Phi_\infty : a \mapsto \sqrt{N^-}j, b \mapsto \sqrt{l}i.$$

Note that we can identify the subalgebra of elements of the form $a + bi$ in $\mathbb{H}$ with the field $\mathbb{C}$ of complex numbers and $\mathbb{H} = \mathbb{C} + \mathbb{C}j$. We fix an isomorphism $\rho : \mathbb{H} \otimes_{\mathbb{R}} \mathbb{C} \simeq M_2(\mathbb{C})$ characterized by

$$\rho(\gamma + \delta j) = \begin{pmatrix} \gamma & \delta \\ -\overline{\delta} & \overline{\gamma} \end{pmatrix}$$

for $\gamma, \delta \in \mathbb{C}$. We denote by the same symbol ρ the composite map $(\rho \otimes 1) \circ \Phi_\infty : B_\infty^\times \to \mathrm{GL}_2(\mathbb{C})$. Let F be the subfield of $\mathbb{C}$ generated by $\sqrt{N^-}$ and $\sqrt{-l}$ and $R_0 = \mathcal{O}_{F,(l)}$ the subring of F obtained from $\mathcal{O}_F$ by inverting l. Then one checks immediately that

$$\rho(\mathcal{O}') \subset M_2(R_0).$$

We consider B as a quadratic space over $\mathbb{Q}$, the quadratic form being the reduced norm. Let $\mathrm{GO}(B)$ denote the corresponding orthogonal similitude group. One has a surjection $\kappa : B^\times \times B^\times \to \mathrm{GO}(B)^0$ onto the connected component of $\mathrm{GO}(B)$, given by $(\gamma_1, \gamma_2) \mapsto (x \rightsquigarrow \gamma_1 x \gamma_2^{-1})$, the kernel being a copy of $\mathbb{G}_m$ embedded diagonally. Then there are theta lifts

$$\Theta(\cdot, \psi) : \mathcal{A}_0(G) \to \mathcal{A}_0(G')$$
$$\Theta^t(\cdot, \psi) : \mathcal{A}_0(G') \to \mathcal{A}_0(G)$$

for $G = \mathrm{GL}_2, G' = \mathrm{GO}(B)^0$ (see [6], [7], and [18] for more details). Note that via κ, automorphic representations of G' can be identified with pairs (π_1, π_2), the π_i being representations of $B^\times$ such that $\xi_{\pi_1} \cdot \xi_{\pi_2} = 1$. Here ξ_{π_i} denotes the central character of π_i.

Let π_B denote an automorphic cuspidal representation of $B^\times$, $\overline{\pi}_B$ its complex conjugate and set $\pi = \mathrm{JL}(\pi_B)$.

Theorem 3.1 (Shimizu).

1. $\Theta(\pi, \psi) = \pi_B \times \overline{\pi}_B$.
2. $\Theta^t(\pi_B \times \overline{\pi}_B, \psi) = \pi$.

Suppose now that π corresponds to a holomorphic newform f of weight $2k$ on $\Gamma_0(N)$ with N square-free. We assume that the first Fourier coefficient of f is equal to 1 and denote by the same symbol f the corresponding adelic automorphic form. On $B^\times$, there is no theory of q-expansions and it is not clear how one might pick a canonical element of π_B analogous to the element f in π. However, the situation can be partially remedied as follows. The representation π_B is a restricted tensor product $\pi_B \simeq \otimes_v \pi_{B,v}$ of local representations. For finite primes $v = q$ such that B is split at q, let g_q be a local new vector in $\pi_{B,q}$ as given by Casselman's theorem, i.e., g_q is nonzero and invariant under the action of

$$\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}_q), c \in N\mathbb{Z}_q \right\}.$$

Here we have identified $(B \otimes \mathbb{Q}_q)^\times$ with $\mathrm{GL}_2(\mathbb{Q}_q)$ via the isomorphism Φ_q. For finite primes v such that B is ramified at v, the local representation $\pi_{B,v}$ is one-dimensional since π_v is a special representation. In this case, we pick g_v to be any nonzero vector in $\pi_{B,v}$. Finally for $v = \infty$, there are two cases since B is split or ramified at infinity. In the former case, we pick g_∞ to be the unique nonzero vector up to scaling on which $\kappa_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ acts by $e^{2ik\theta}$. In the latter case, one has that the representation $\pi_{B,\infty}$ is isomorphic to

$$\rho_k : B_\infty^\times \to \mathrm{GL}_{2k-1}(\mathbb{C}), \qquad \rho_k = \mathrm{Sym}^{2k-2} \rho \otimes (\det \rho)^{1-k}.$$

Let $V_1 = \mathbb{C}^2$ be the representation space associated to ρ and denote by e_1, e_2 the standard basis. Then the set of vectors $e_1^{\otimes r} \otimes e_2^{\otimes 2k-2-r}, 0 \le r \le 2k-2$ is a basis for V_k, the representation space of ρ_k. Fixing an isomorphism between $\pi_{B,\infty}$ and V_k, we pick $g_\infty^l = e_1^{\otimes r} \otimes e_2^{\otimes 2k-2-r}$. Notice that g_∞^r spans the unique line in $\pi_{B,\infty}$ on which $e^{i\theta} \in \mathbb{C}^{(1)}$ acts by $e^{2i(r-(k-1))\theta}$. Thus if B is indefinite, $g_B = \otimes_v g_v$ in π_B is well-defined up to scaling, while if B is definite, the vector of forms $[g_B^r]$, with $g_B^r = \otimes_{v<\infty} g_v \otimes g_\infty^r$ in π_B is well-defined up to scaling. We will see below that for a given prime p, we can pick g (resp. $[g_B^r]$) in such a way that it is well-defined up to a p-adic unit in K_f.

We will now pick a Schwartz function φ (resp. functions φ^r) in $\mathcal{S}(B_{\mathbb{A}})$ such that $\theta(f, \varphi, \psi) = \beta g_B$ (resp. $[\theta(f, \varphi^l, \psi)] = \beta[g_B^r]$) in the indefinite (resp. definite) case for some scalar β. Suppose that $N = N^+N^-$ and let $\mathcal{O}$ be the unique Eichler order of level N^+ in B such that

$$\Phi_q(\mathcal{O} \otimes \mathbb{Z}_q) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}_q), c \in N^+\mathbb{Z}_q \right\} \text{ for } q \nmid N^- .$$

Note that for $q \mid N^-$, $\mathcal{O} \otimes \mathbb{Z}_q$ is just the unique maximal order in $B \otimes \mathbb{Q}_q$. Now for $v = q$ a finite prime, set $\varphi_q =$ the charactersitic function of $\mathcal{O} \otimes \mathbb{Z}_q$. If $v = \infty$ and B is indefinite, set

$$\varphi_\infty(\beta) = \frac{1}{\pi} Y(\beta)^k e^{-2\pi(|X(\beta)|^2 + |Y(\beta)|^2)}$$

where for $\beta = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{R})$, $X(\beta) = \frac{1}{2}(a+d) + \frac{i}{2}(b-c)$ and $Y(\beta) = \frac{1}{2}(a-d) + \frac{i}{2}(b+c)$. As usual we have identified $B \otimes \mathbb{R}$ in this case with $M_2(\mathbb{R})$ via Φ_∞. If B is definite, Φ_∞ identifies $B \otimes \mathbb{R}$ with the space of Hamilton quaternions. Set

$$\begin{aligned} \varphi_\infty^r(u + vj) &= \bar{v}^{2l} p_{k-1-|l|}(|u|^2) e^{-2\pi(|u|^2+|v|^2)}, \text{ if } l \geq 0 \\ &= v^{2|l|} p_{k-1-|l|}(|u|^2) e^{-2\pi(|u|^2+|v|^2)}, \text{ if } l \leq 0 \end{aligned}$$

where $l = k - 1 - r$ and p_m is the Laguerre polynomial of degree m, given by

$$p_m(t) = \sum_{j=0}^{m} \binom{l}{j} \frac{(-t)^j}{j!}.$$

Finally, set $\varphi^r = \otimes_v \varphi_v \otimes \varphi_\infty^r$. The following proposition follows easily from computations in [32] and [34].

Proposition 3.2. *Suppose that B is indefinite (resp. definite.) Let $\delta := \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \in B_\infty^\times$ (resp. $\delta := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in B_\infty^\times$.) Then*

$$\theta(f, \varphi, \psi)(x \cdot \delta) = \overline{\theta(f, \varphi, \psi)(x)}.$$

Further, there exist nonzero scalars α, β (resp. α_r, β) such that

(a) $\theta(f, \varphi, \psi) = \beta \cdot (g_B \times g_B)$ (resp. $[\theta(f, \varphi^r, \psi)] = \beta \cdot [g_B^r \times g_B^r]$).
(b) $\theta^t(g_B \times g_B, \varphi, \psi) = \alpha f$ (resp. $\theta^t(g_B^r \times g_B^r, \varphi^r, \psi) = \alpha_r f$).

Note that by our assumption that f occurs on $\Gamma_0(N)$, π and π_B both have trivial central character, hence $\overline{\pi}_B = \pi_B$. If B is indefinite (resp. definite) let Ψ (resp. $\tilde{\Psi}$) be such that $\theta(f, \varphi, \psi) = \Psi \times \Psi$ (resp. $[\theta(f, \varphi^r, \psi)] = \tilde{\Psi} \times \tilde{\Psi}$.)

In the following discussion we write $\theta(f)$ instead of $\theta(f, \varphi, \psi)$ for simplicity of notation. There are two important formulas that are very useful in this situation, namely see-saw duality ([13] and [7]) and the Rallis inner product formula. In the indefinite case, applying see-saw duality gives

$$\begin{aligned} \langle \theta(f), g_B \times g_B \rangle &= \langle f, \theta^t(g_B \times g_B) \rangle \\ \beta \langle g_B, g_B \rangle^2 &= \overline{\alpha} \langle f, f \rangle \end{aligned} \tag{3.1}$$

where $\langle\ ,\ \rangle$ is the Petersson inner product. Next the Rallis inner product formula gives

$$\begin{aligned}\langle \theta(f), \theta(f)\rangle &\sim L(1, ad^0(\pi))\langle f, f\rangle \\ \beta\overline{\beta}\langle g_B, g_B\rangle^2 &\sim \langle f, f\rangle^2\end{aligned} \tag{3.2}$$

Combining (3.1) and (3.2) yields

$$\overline{\alpha}\overline{\beta} \sim \langle f, f\rangle \tag{3.3}$$

and

$$\alpha\overline{\alpha} \sim \langle g_B, g_B\rangle^2. \tag{3.4}$$

Clearly, exactly the same formulas hold also in the definite case, with α, β, g_B being replaced by α_r, β, g_B^r respectively. In particular from (3.3) we see that $\overline{\alpha_r} \sim \overline{\alpha_{r'}}$ and hence $\alpha_r \sim \alpha_{r'}$. This implies also that $\langle g_B^r, g_B^r\rangle \sim \langle g_B^{r'}, g_B^{r'}\rangle$.

The indefinite case

In this section we suppose that B is indefinite. The form g_B that we picked in the previous section corresponds in the usual way to a classical modular form on the upper half plane $\mathfrak{H}$ (which we denote simply by the symbol g) with respect to the group $\mathcal{O}^{(1)}$ consisting of the elements of $\mathcal{O}$ with reduced norm. Further we may view $\varsigma = g(z)(2\pi i \cdot dz)^{\otimes k}$ as being a section of the line bundle Ω^k on the curve $X = \mathfrak{H}/\mathcal{O}^{(1)}$. One knows from the work of Shimura that the curve X admits a canonical model $X_{\mathbb{Q}}$ over $\mathbb{Q}$. Let $\mathcal{X}$ denote the minimal regular model of X over $\mathcal{O}_{K_f}$ and denote by ω the relative dualizing sheaf on $\mathcal{X}/\operatorname{spec}\mathcal{O}_{K_f}$. Since the Hecke eigenvalues of g lie in K_f, we may choose g such that $\varsigma \in H^0(X_{K_f}, \Omega^k)$ and ς is a p-unit in $H^0(\mathcal{X}, \omega^k)$. Thus g is well-defined up to a p-unit in K_f. Fixing such a choice of g, one has:

Theorem 3.3 (Harris–Kudla [6]). *$\beta \in K_f$. Consequently $\langle f, f\rangle/\langle g, g\rangle \in K_f$.*

Indeed, since K_f is totally real, one gets from (3.2) that $\beta \cdot \langle g, g\rangle \sim \langle f, f\rangle$ so that

$$\beta \sim \frac{\langle f, f\rangle}{\langle g, g\rangle}.$$

We are now in a situation where Questions A and B of the introduction make sense, namely, we can ask for information about $v_\lambda(\beta)$ for λ a prime in K_f above p. The answer is provided by the following theorem and corollary which constitute the main results of [18].

Theorem 3.4. *(a)*

$$v_\lambda(\beta) = \min_{K,\chi} v_\lambda(\delta(\pi, K, \chi))$$

where

$$\delta(\pi, K, \chi) := \frac{L(\frac{1}{2}, \pi_K \otimes \chi)}{\Omega_K^{4k}}.$$

Here K ranges over imaginary quadratic fields that are split at N^+ and inert at N^-, χ ranges over unramified Hecke characters of K of type $(k, -k)$ at infinity, and Ω_K is a suitable CM period, i.e., a period of a Neron differential on an elliptic curve that has CM by $\mathcal{O}_K$.

(b)

$$v_\lambda(\delta(\pi, K, \chi)) \geq 0$$

for all K, χ as above. Further if there exists a newform f' of level M dividing N but not divisible by N^- such that $\rho_f \equiv \rho_{f'} \mod \lambda$, then $v_\lambda(\delta(\pi, K, \chi)) > 0$.

(c) $v_\lambda(\beta) \geq 0$. Further, if there exists a newform f' of level M dividing N but not divisible by N^- such that $\rho_f \equiv \rho_{f'} \mod \lambda$, then $v_\lambda(\beta) > 0$.

The reader will note that part (c) of the theorem follows immediately from parts (a) and (b). We first indicate briefly some of the ingredients in the proof of part (a). For K, χ as above, pick a *Heegner* embedding $K \hookrightarrow B$ and set

$$L_\chi(g_B) = j(\alpha, i)^{2k} \int_{K^\times \backslash K_{\mathbb{A}}^\times / K_\infty^\times} g_B(x\alpha)\chi(x) d^\times x$$

for any $\alpha \in \mathrm{SL}_2(\mathbb{R})$ such that $\alpha^{-1} \cdot (K \otimes \mathbb{R}) \cdot \alpha = \left\{ \begin{pmatrix} a & -b \\ b & a \end{pmatrix}, a, b \in \mathbb{R} \right\}$. (We note that such Heegner embeddings exist, if and only if, K is split at the primes dividing N^+ and inert at the primes dividing N^-.) For a suitable choice of measure on $K_{\mathbb{A}}^\times$, the integral above can be interpreted as a sum of values of g at certain CM points associated to K, twisted by the values of χ, and divided by the class number h_K. Now, a λ-integral modular form must take λ-integral values at CM points up to suitable CM periods. Conversely, any form that takes on λ-integral values for a large set of CM points must be λ-integral. Hence one can show roughly that

$$\min_{K,\chi,p \nmid h_K} v_\lambda \left(\frac{L_\chi(g_B)}{\Omega_K^{2k}} \right) = 0. \tag{3.5}$$

On the other hand, by the methods of Waldspurger one can show that

$$\beta L_\chi(g_B)^2 = L_{\chi \times \chi}(\theta(f)) \sim \frac{1}{h_K^2} L(\frac{1}{2}, \pi_K \otimes \chi) \tag{3.6}$$

Part (a) of the theorem follows now by combining (3.5) and (3.6).

Next, we give a brief outline of the proof of part (b). We first assume that p is split in K, and $p \nmid h_K$. By the Rankin–Selberg method

$$L(\frac{1}{2}, \pi_K \otimes \chi) = \langle fE, \theta_\chi \rangle$$

where E is a certain weight-1 Eisenstein series and $\theta_\chi \in S_{2k+1}(\Gamma_1(d_K))$ is the theta function associated to χ. The form fE has integral Fourier coefficients, since E does. Let us expand

$$fE = \gamma \cdot \theta_\chi + H$$

where H is a linear combination of forms orthogonal to θ_χ. Then

$$\frac{\langle fE, \theta_\chi \rangle}{\Omega_K^{4k}} = \gamma \frac{\langle \theta_\chi, \theta_\chi \rangle}{\Omega_K^{4k}} \sim \gamma \cdot \frac{L(\chi(\chi^\rho)^{-1}, 1)}{\Omega_K^{4k}}$$

where $\chi^\rho = \chi \circ \rho$ is the twist of χ by complex conjugation ρ. From Shimura, one knows that both γ and $L(\chi) := L(\chi(\chi^\rho)^{-1}, 1)/\Omega_K^{4k}$ are algebraic; in fact one even knows, for example from the results in [22] that $L(\chi)$ is λ-integral. The problem is that γ is unlikely to be λ-integral. However if γ had a denominator, by multiplying (3.7) by an appropriate power of λ, we would get congruences modulo λ between θ_χ and other forms orthogonal to θ_χ. Let us assume for the moment that $\theta_\chi \equiv h \mod \lambda$ for some eigenform h that is not a theta lift from K. On the one hand, the λ-adic representation $\rho_{h,\lambda}$ associated to h is irreducible even when restricted to $\mathrm{Gal}(\overline{K}/K)$ since h is not a theta lift from K; on the other hand, $\overline{\rho}_{h,\lambda}|_{\mathrm{Gal}(\overline{K}/K)}$ is reducible, being isomorphic mod λ to $\chi_\lambda \oplus \chi_\lambda^\rho = \rho_{\theta_\chi,\lambda}|_{\mathrm{Gal}(\bar{K}/K)}$, where $\chi_\lambda, \chi_\lambda^\rho$ denote the λ-adic characters associated to χ, χ^ρ, respectively. This latter fact can be used to construct a lattice in the representation space of $\rho_{h,\lambda}$ whose reduction modulo λ is an extension of χ_λ^ρ by χ_λ. For simplicity, let us also say that the class number of K is one. If K_∞ is the unique $\mathbb{Z}_p^2$ extension of K, the splitting field of the extension obtained above is an abelian p-extension K' of K_∞ with controlled ramification such that the conjugation action of $\mathrm{Gal}(\overline{K}/K)$ on $\mathrm{Gal}(K'/K_\infty)$ is via the character $\chi_\lambda(\chi_\lambda^\rho)^{-1}$.

The idea that one can construct extensions by the method above is originally due to Ribet [19]; however we need to employ the more refined methods of Wiles [33]. The upshot is that we can construct an extension K'/K_∞, as above, whose size is at least as large as the denominator of γ. In contrast, the main conjecture of Iwasawa theory for K (a theorem of Rubin [21]) can be used to bound the size of such an extension from above by the L-value $L(\chi)$. Thus any possible denominators in γ are cancelled by the numerator of $L(\chi)$ and the product $\gamma \cdot L(\chi)$ is λ-integral as desired. We refer the reader to Chapter 4 of [18] for more details on the above constructions and to Section 5.3 of the same article for the proof that the L-values above are divisible by the expected level-lowering congruence primes.

Remark 3.5. The proof outlined above works whenever p is split in K and $p \nmid h_K$. This is enough to conclude part (c) of the theorem, since one has infinitely many CM points satisfying these conditions. But by parts (a) and (c), we see that part (b) must remain true even is p is inert in K or $p \mid h_K$ (or both.) Thus the *ordinary* case of part (b) is used to prove the *supersingular* case of the same.

Remark 3.6. In fact, one does not need the full main conjecture to deduce the above integrality result but only the *anticyclotomic* part. It is also sometimes possible to prove directly the integrality of $\langle G, \theta_\chi \rangle / \Omega_K^{4k}$ for G in integral form and then deduce the anticyclotomic main conjecture as a consequence. Indeed, this is the approach taken by Tilouine [28] and Hida [12]. The latter article deals also with the case of CM fields. However, the results of these articles require extra hypotheses that may not always be satisfied in our situation. Thus it seems to the author that only the approach above — using the main conjecture rather than deducing it as a consequence — provides the requisite precision needed in our analysis.

Remark 3.7. One would certainly expect conversely, that if $v_\lambda(\langle f, f\rangle / \langle g, g\rangle) > 0$, then λ is a level-lowering congruence prime of the expected type. One might expect an even stronger statement to be true, namely a canonical factorization of this ratio as a product of integers, the terms of the product being indexed by the primes dividing N^- and admitting a geometric interpretation, as is the case for elliptic curves (see Example 2.3). Unfortunately, we have nothing to say about this problem at present for forms of higher weight.

The definite case

In this section, we suppose that B is a definite quaternion algebra. Recall that for every integer r satisfying $0 \le r \le (2k-2)$, we have picked a form g_B^r on $B_{\mathbb{A}}^\times$ such that the vector of forms $[g_B^r]$ is well-defined up to a scalar. Set $\tilde{g}_B = [g_B^r]^t$, so that $\tilde{g}_B \in \tilde{S}_k(U)$ where

$$\tilde{S}_k(U) = \{\tilde{g} : B^\times \setminus B_{\mathbb{A}}^\times \to \mathbb{C}^{2k-1} \mid \tilde{g}(x \cdot u u_\infty) = \rho_k(u_\infty)\tilde{g}(x) \ \forall \ u \in U, u_\infty \in B_\infty^\times\}$$

and $U = \prod_q U_q$ is the open compact subgroup of $B_{\mathbb{A}_f}^\times$ given by $U_q = (\mathcal{O} \otimes \mathbb{Z}_q)^\times$.

Since any element of $\tilde{S}_k(U)$ is determined by its values on $B_{\mathbb{A}_f}^\times$, the space $\tilde{S}_k(U)$ is canonically isomorphic to the space $S_k(U)$ given by

$$S_k(U) = \{g : B_{\mathbb{A}_f}^\times / U \to \mathbb{C}^{2k-1} \mid g(\alpha \cdot x) = \rho_k(\alpha^{-1}) g(x) \ \forall \alpha \in B^\times\}.$$

Denote by g_B the element of $S_k(U)$ corresponding to $\tilde{g}_B$. We now follow [27] in defining an integral (or rather p-integral) structure on $S_k(U)$. For R

any ring such that $R_0 \subset R \subset \mathbb{C}$, let $L_k(R)$ be the R-submodule of $\mathbb{C}^{2k-1}$ consisting of vectors all of whose components are in R. The group $B^\times_{\mathbb{A}_f}$ acts on R_0-lattices in $L_k(K)$ via the embedding

$$B^\times_{\mathbb{A}_f} \hookrightarrow (B \otimes \mathbb{A}_{K,f})^\times \xrightarrow{\mu_k \otimes 1} \mathrm{GL}_{2k-1}(\mathbb{A}_{K,f}).$$

Set $L_k(R) \cdot x := L_k(\mathcal{O}_K) \cdot x \otimes R$. We then define $S_k(U;R)$ to be the set of $h \in S_k(U)$ such that $h(x) \in L_k(R) \cdot x^{-1}$ for all $x \in B^\times_{\mathbb{A}_f}$.

Let $K_{0,f} = K_f F$ be the compositum of K_f and F. We may then normalize g_B by requiring that it be a p-unit in $S_k(U, R_p)$ where R_p is the subring of p-integral elements in $K_{0,f}$. With this normalization, it makes sense to study the arithmetic properties of α_r and β. Note that this case is very different from the indefinite case in that $\alpha_r \in \overline{\mathbb{Q}}$ while $\beta/u_+(f)u_-(f) \in \overline{\mathbb{Q}}$.

Let $(,)$ denote the inner product on $S_k(U)$ defined in [27] (and denoted by $\langle , \rangle$ in that article). For $g_B \in S_k(U)$ and $\delta = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $g'_B := \rho_k(\delta)\overline{g} \in S_k(U)$ and it is easy to see that

$$\langle \tilde{g}_B, \tilde{g}_B \rangle = \sum_r \langle g^r_B, g^r_B \rangle = (g_B, g'_B).$$

Note that $\langle f, f \rangle^2 \sim \beta\overline{\beta}\langle \tilde{g}_B, \tilde{g}_B \rangle^2 = \langle \tilde{\Psi}, \tilde{\Psi} \rangle^2 = (\Psi, \Psi)^2 = \beta^2 (g_B, g_B)^2$ since $\overline{\Psi(x) \cdot \delta} = \Psi(x)$ (and hence $\Psi' = \Psi$).

Set $\delta_g = (g_B, g_B)$. As in the indefinite case, one may define an invariant η_g that counts congruences satisfied by g; one always has that $(\eta_g) \subseteq \delta_g$ and in good situations (namely when some freeness condition holds) one has also $(\delta_g) = \eta_g$. Now

$$\beta \sim \frac{\langle f, f \rangle}{\delta_g} \sim \frac{\delta_f}{\delta_g} u_+(f) u_-(f)$$

and $\alpha \sim (g_B, g_B)$. Since $\eta_g \subseteq \eta_f$, we obtain:

Theorem 3.8. *(a)* $v_\lambda(\alpha) \geq 0$.
(b) $v_\lambda\left(\frac{\beta}{u_+(f)u_-(f)}\right) = v_\lambda\left(\frac{\delta_f}{\delta_g}\right)$. *In particular, if* $(\delta_f) = \eta_f$,

$$v_\lambda\left(\frac{\beta}{u_+(f)u_-(f)}\right) \geq 0.$$

We now explain the relation between this result and Rankin–Selberg L-values. Let K be an imaginary quadratic field that is split at N^+ and inert at N^-, $i : K \hookrightarrow B$ is a Heegner embedding with $p \nmid h_K$. For any integer r with $0 \leq r \leq 2k-2$, let χ_r be an unramified Hecke character of K of type $(r_0, -r_0)$ at infinity, where $r_0 = r - (k-1)$. With a suitable choice of measure, one defines

$$L_{\chi_r}(g_B^r) = \int_{K^\times \backslash K_{\mathbb{A}}^\times / K_\infty^\times} g_B^r(x\gamma)\chi(x)d^\times x$$

for any $\gamma \in (B \otimes \mathbb{R})^\times = \mathbb{H}^\times$ such that $\gamma^{-1}(K \otimes \mathbb{R})\gamma = \mathbb{C} \subset \mathbb{H}$. Again, by methods of Waldspurger one can prove that

$$\beta L_{\chi_r}(g_B^r)^2 = L_{\chi_r \times \chi_r}(\theta(f, \varphi_r)) \sim L(\frac{1}{2}, \pi_K \otimes \chi_r).$$

Combining this with (3.2) yields

$$|L_{\chi_r}(g_B^r)|^2 \sim L(\frac{1}{2}, \pi_K \otimes \chi_r)\frac{\langle g_B^r, g_B^r \rangle}{\langle f, f \rangle}$$

which is just one form of Gross's special value formula.

The following integrality criterion for forms on $B^\times$ follows quite easily from the equidistribution theorem (Theorem 10) of [14].

Proposition 3.9 (Integrality criterion for forms on $B^\times$). *A form $\tilde{\Psi}' = [\Psi_r']$ is p-integral if and only if for some Heegner point $K \hookrightarrow B$ with $p \nmid h_K$ and $h_K >> 0$, and all unramified characters χ_r of $K_{\mathbb{A}}^\times$ of infinity-type $(r_0, -r_0)$, $0 \leq r \leq 2k-2$,*

$$L_{\chi_r}(\Psi_r') := \int_{K^\times K_\infty^\times \backslash K_{\mathbb{A}}^\times} \Psi_r'(x)\chi_r(x)d^\times x \qquad (3.7)$$

is p-integral.

Note that the expression (3.7) is a finite sum of the values Ψ_r' twisted by the values of the character χ_r. Applying the criterion to the form $\tilde{\Psi}$ constructed earlier and using Theorem 3.8, we see that

Theorem 3.10. *For K, r, χ_r as above, and λ any prime above p,*

$$v_\lambda\left(\frac{L(\frac{1}{2}, \pi_K \times \chi_r)}{u_+(f)u_-(f)}\right) \geq v_\lambda(\frac{\delta_f}{\delta_g}).$$

Further, for any K with $h_K >> 0$, there exists a pair (r, χ_r) such that equality holds.

3.2 The dual pair $(\widetilde{\mathbf{SL_2}}, \mathbf{O}(V))$, $V = B^0$

The indefinite case

Note that Theorem 3.4 above does not address the periods $u_\pm(f), u_\pm(g)$; rather, it pertains only to the Petersson norms which are products of periods. It turns out to be much harder to prove results about individual periods. By studying a relevant theta correspondence, we are able to prove the following result about ratios of periods. We denote by $A(f,d)$ the algebraic part of the L-value $L(\frac{1}{2},\pi_f\otimes\chi_d)$ for d any fundamental quadratic discriminant and χ_d the corresponding character, i.e., $A(f,d)=\mathfrak{g}(\chi_d)L(\frac{1}{2},\pi_f\otimes\chi_d)/u_\tau(f)$ for $\tau=(-1)^k\mathrm{sign}(d)$. It is known that $A(f,d)\in K_f$ and that it is a p-adic integer at least when p is not an Eisenstein prime for f.

Theorem 3.11. *Suppose that N is odd and square-free.*

(a) Let $\sigma\in Aut(\mathbb{C}/\mathbb{Q})$. Then

$$\left(\frac{u_\pm(f)}{u_\pm(g)}\right)^\sigma=\frac{u_\pm(f^\sigma)}{u_\pm(g^\sigma)}.$$

(b) Suppose there exists a quadratic discriminant d such that $p\nmid A(f,d)$. Then

$$v_\lambda\left(\frac{u_\tau(f)}{u_\tau(g)}\right)\geq 0$$

where $\tau=(-1)^k\mathrm{sign}(d)$.

In the case f has weight 2, one can use the rationality of period ratios provided by part (a) to construct directly isogenies between quotients of $J_0(N)$ and $Jac(X)$, completely independent of Faltings' isogeny theorem. Further in the case when $k=2$, one also has applications relating to questions about p-divisibility and the indivisibility of central values of quadratic twists. (See [15] and [17] for more details on these applications.)

The proofs of the above results are based on studying the p-adic properties of the theta lifting for the dual pair $(\widetilde{\mathrm{SL}}_2,\mathrm{O}(V))$ with V the space of trace zero elements in B. The automorphic theory in this case has been worked out in great detail in three beautiful articles of Waldspurger ([29], [30] and [31]). In the arithmetic theory there are three complications that arise. Firstly, there is not one automorphic form on $\widetilde{\mathrm{SL}}_2$ but rather a *packet* of forms that corresponds to g_B. Secondly, there is no good theory of newforms for forms of half-integral weight. Lastly, while one can again measure arithmeticity on $PB^\times=\mathrm{SO}(V)$ by means of period integrals on tori, the relevant period integrals are not related to a Rankin–Selberg L-value as in the case of $\mathrm{O}(B)$. However, for a suitable choice of ψ and $\varphi\in V(\mathbb{A})$ and a suitable form h that has weight $k+\frac{1}{2}$ and that is p-adically normalized, one can show:

Theorem 3.12. *(a) $\theta^t(g,\varphi,\psi)=\alpha u_\pm(g)h$ for some scalar α.*
(b) $\theta(h,\varphi,\psi)=\beta g$ for some scalar β.
(c) $\alpha,\beta\in\overline{\mathbb{Q}}$. Further $v_\lambda(\alpha)\geq 0$ and $v_\lambda(\beta)\geq 0$.

The proof of the above theorem (especially the p-integrality of β) is rather intricate, so we refer the reader to the article [15] for more details.

The definite case

It is not hard to show in this case that for suitable choices of φ, ψ, and h, $\theta^t(g,\varphi,\psi) = \alpha h$ and $\theta(h,\varphi,\psi) = \beta u_\pm(f)g$ for some scalars α,β. Unfortunately, the author does not know how to prove in this case the analog of Theorem 3.8(b), i.e., the p-integrality of β. One would certainly conjecture that:

Conjecture 3.13. $v_\lambda\left(\frac{\beta}{u_\pm(f)}\right) \geq 0$.

However the previous methods of proofs break down; one seems to require rather refined information, about Petersson inner products and congruences of half-integral weight forms, that is not presently available. More precisely, one is lead to conjecture that the algebraic parts of Petersson inner products of half-integral weight forms count congruences satisfied by these forms just as in the integral weight setting. The reader is referred to [17] for a discussion of this issue.

4 A conjecture on Petersson inner products of quaternionic modular forms over totally real fields

In this section, we consider conjectural generalizations to Hilbert modular forms and their quaternionic analogs over totally real fields. Let F be a totally real field, $\Sigma_{F,\infty}$ (resp. $\Sigma_{F,\mathrm{fin}}$, resp. Σ_F) the set of infinite places (resp. finite places, resp. all places) of F. Let $\pi = \otimes_v \pi_v$ the automorphic representation of $\mathrm{GL}_2(\mathbb{A}_F)$ corresponding to a holomorphic Hilbert modular form f with even weights at infinity. We will assume for simplicity that π_v is a special representation for all finite places v of F such that π_v is discrete series. Let B be a quaternion algebra over F such that π admits a Jacquet–Langlands transfer π_B to B, i.e., such that for all places v where B is ramified, π_v is a discrete series representation. As explained before, we can pick a non-zero element $g_B \in \pi_B$ that is well-defined up to multiplication by a scalar. We suppose that g_B is arithmetically normalized and consider the Petersson inner product $\langle g_B, g_B\rangle$.

Remark 4.1. The form g_B corresponds to a section of an *automorphic* vector bundle V on a Shimura variety X_B attached to B. Such vector bundles are known to have canonical models over specified number fields based on the work of Harris [4]. Thus it is perfectly clear how to normalize g_B up to an element in a specific number field. However, to normalize g_B up to a p-adic unit, one needs to construct canonical integral models of X and V over suitable p-adic rings. We will assume in what follows that such models exist, and hence that g_B may be normalized up to a p-adic unit.

The problem of relating the numbers $\langle g_B, g_B\rangle$ as B varies was first considered by Shimura in the early 80's. Shimura proved that up to algebraic factors, this Petersson inner product only depends on the set of infinite places at which B is unramified. Further, he conjectured that to each infinite place v of F, one can associate a (transcendental) number c_v depending only on π, such that

$$\langle g_B, g_B\rangle \sim_{\overline{\mathbb{Q}}^*} \prod_{\substack{v\in\Sigma_{F,\infty}\\ v\nmid \mathrm{disc}B}} c_v. \tag{4.1}$$

This conjecture was proved by Harris ([5]) under the hypothesis that for at least one finite place v, π_v is discrete series. Notice that (4.1) implies that

$$\langle f, f\rangle \sim_{\overline{\mathbb{Q}}^*} \prod_{v\in\Sigma_{F,\infty}} c_v$$

and thus

$$\langle g_B, g_B\rangle \sim_{\overline{\mathbb{Q}}^*} \frac{\langle f, f\rangle}{\prod_{\substack{v\in\Sigma_{F,\infty}\\ v\mid \mathrm{disc}B}} c_v}.$$

On the other hand, by Example 2.3, for forms corresponding to elliptic curves over $\mathbb{Q}$, with B an indefinite quaternion algebra, we have

$$\langle g_B, g_B\rangle \sim \frac{\langle f, f\rangle}{\prod_{\substack{v\in\Sigma_{F,\mathrm{fin}}\\ v\mid \mathrm{disc}B}} c_v}$$

where $\sim$ now denotes equality up to p-units, and the c_v's are orders of certain component groups. This leads us to make the following conjecture in the totally real case.

Conjecture 4.2. *Suppose that p is a generic prime for (F, π), i.e., p is prime to $\mathrm{disc}_{F/\mathbb{Q}}$, $\deg_{F/\mathbb{Q}}$, the class number of F, and the level of π. Then for each place v of F such that π_v is a discrete series, there exists a complex number c_v such that*

$$\langle g_B, g_B\rangle \sim \frac{\langle f, f\rangle}{\prod_{\substack{v\in\Sigma_F\\ v\mid \mathrm{disc}B}} c_v}$$

where $\sim$ denotes equality up to a p-unit.

Remark 4.3. One would expect the c_v's to be transcendental for v infinite and algebraic integers for v finite. Further for finite v, c_v should measure level-lowering congruences at v satisfied by π. Note that the c_v's are not local invariants, i.e., they are not determined by the local representation π_v but rather depend very much on the global representation π.

Remark 4.4. It is well-known that $\langle f, f\rangle \sim L(\mathrm{ad}^0\pi, 1)$ where $\mathrm{ad}^0\pi$ denotes the adjoint representation. It is an interesting problem, suggested to the author by Colmez, to study the relation between the conjecture above and the Bloch–Kato conjecture for the adjoint L-value. We hope to take this up in future work.

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Functoriality and Special Values of L-Functions

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Summary. This is a semi-expository article concerning Langlands functoriality and Deligne's conjecture on the special values of L-functions. The emphasis is on symmetric power L-functions associated to a holomorphic cusp form, while appealing to a recent work of Mahnkopf on the special values of automorphic L-functions.

1 Introduction

Langlands functoriality principle reduces the study of automorphic forms on the adèlic points of a reductive algebraic group to those of an appropriate general linear group. In particular, every automorphic L-function on an arbitrary reductive group must be one for a suitable GL_n. One should therefore be able to reduce the study of of special values of an automorphic L-function to those of a principal L-function of Godement and Jacquet on GL_n.

While the integral representations of Godement and Jacquet do not seem to admit a cohomological interpretation, the recent work of J. Mahnkopf ([28] and [29]) provides us with such an interpretation for certain Rankin–Selberg type integrals. In particular, modulo a nonvanishing assumption on local archimedean Rankin–Selberg product L-functions for forms on $\mathrm{GL}_n \times \mathrm{GL}_{n-1}$, he defines a pair of periods, which seem to be in accordance with those of Deligne [9] and Shimura [43]. This work of Mahnkopf is quite remarkable and requires the use of both Rankin–Selberg and Langlands–Shahidi methods in studying the analytic (and arithmetic) properties of L-functions. His work therefore brings in the theory of Eisenstein series to play an important role. In Section 6 we briefly review this work of Mahnkopf.

This article is an attempt to test the philosophy–to study the special values of L-functions while using functoriality–by means of recent cases of functoriality established for symmetric powers of automorphic forms on GL_2 ([17] and [22]). While a proof of the precise formulas in the conjectures of Deligne [9] still seem to be out of reach, we expect to be able to prove explicit connections between the special values of symmetric power L-functions twisted by Dirichlet characters and those of the original symmetric power L-functions using this work of Mahnkopf. These relations are formulated in this paper as Conjecture 7.1, which also follows from the more general conjectures of Blasius [3] and Panchishkin [33], although the heuristics underlying our conjecture are quite different.

A standard assumption made in the study of special values of L-functions is that the representations (to which are attached the L-functions) are cohomological. This is the case in Mahnkopf's work. A global representation being cohomological is entirely determined by the archimedean components. For representations which are symmetric power lifts of a cusp form on GL_2 we have the following fact. Consider a holomorphic cusp form on the upper half plane of weight k. This corresponds to a cuspidal automorphic representation, which is cohomological if $k \geq 2$, and any symmetric power lift, if cuspidal, is essentially cohomological. See Theorem 5.5. (If the weight $k = 1$ then the representation is not cohomological, and furthermore none of the symmetric power L-functions have any critical points.) In Section 5 we review representations with cohomology in the case of GL_n.

We recall the formalism of Langlands functoriality for symmetric powers in Section 2. We then review Deligne's conjecture for the special values of symmetric power L-functions in Section 3 and give a brief survey as to which cases are known so far. In Section 4 we sketch a proof of the conjecture for dihedral representations. We add that in the dihedral case the validity of the conjecture is known to experts.

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2 Symmetric powers and functoriality

In this section we recall the formalism of Langlands functoriality especially for symmetric powers. We will be brief here as there are several very good expositions of the principle of functoriality; see for instance [8, Chapter 2].

Let F be a number field and let $\mathbb{A}_F$ be its adèle ring. We let π be a cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_F)$, by which we mean that, for some $s \in \mathbb{R}$, $\pi \otimes |\cdot|^s$ is an irreducible summand of

$$L^2_{\mathrm{cusp}}(\mathrm{GL}_2(F)\backslash\mathrm{GL}_2(\mathbb{A}_F), \omega)$$

the space of square-integrable cusp forms with central character ω. We have the decomposition $\pi = \otimes'_v \pi_v$ where v runs over all places of F and π_v is an irreducible admissible representation of $\mathrm{GL}_2(F_v)$.

The local Langlands correspondence for GL_2 (see [26] and [24] for the p-adic case and [23] for the archimedean case), says that to π_v is associated a representation $\sigma(\pi_v) : W'_{F_v} \to \mathrm{GL}_2(\mathbb{C})$ of the Weil–Deligne group W'_{F_v} of F_v. (If v is infinite, we take $W'_{F_v} = W_{F_v}$.) Let $n \geq 1$ be an integer. Consider the n-th symmetric power of $\sigma(\pi_v)$ which is an $(n+1)$-dimensional representation. This is simply the composition of $\sigma(\pi_v)$ with $\mathrm{Sym}^n : \mathrm{GL}_2(\mathbb{C}) \to \mathrm{GL}_{n+1}(\mathbb{C})$. Appealing to the local Langlands correspondence for GL_{n+1} ([14], [15], [23], and [24]) we get an irreducible admissible representation of $\mathrm{GL}_{n+1}(F_v)$ which we denote as $\mathrm{Sym}^n(\pi_v)$. Now define a global representation of $\mathrm{Sym}^n(\pi)$ of $\mathrm{GL}_{n+1}(\mathbb{A}_F)$ by

$$\mathrm{Sym}^n(\pi) := \otimes'_v \mathrm{Sym}^n(\pi_v).$$

Langlands principle of functoriality predicts that $\mathrm{Sym}^n(\pi)$ is an automorphic representation of $\mathrm{GL}_{n+1}(\mathbb{A}_F)$, i.e., it is isomorphic to an irreducible subquotient of the representation of $\mathrm{GL}_{n+1}(\mathbb{A}_F)$ on the space of automorphic forms [6, §4.6]. If ω_π is the central character of π then $\omega_\pi^{n(n+1)}$ is the central character of $\mathrm{Sym}^n(\pi)$. Actually it is expected to be an isobaric automorphic representation. (See [8, Definition 1.1.2] for a definition of an isobaric representation.) The principle of functoriality for the n-th symmetric power is known for $n = 2$ by Gelbart–Jacquet [11]; for $n = 3$ by Kim–Shahidi [22]; and for $n = 4$ by Kim [17]. For certain special forms π, for instance, if π is dihedral, tetrahedral, octahedral or icosahedral, it is known for all n (see [18] and [34]).

3 Deligne's conjecture for symmetric power L-functions

Deligne's conjecture on the special values of L-functions is a conjecture concerning the arithmetic nature of special values of motivic L-functions at critical points. The definitive reference is Deligne's article [9]. We begin by introducing the symmetric power L-functions, which are examples of motivic L-functions, and then state Deligne's conjecture for these L-functions.

3.1 Symmetric power L-functions

Let $\varphi \in S_k(N, \omega)$, i.e., φ is a holomorphic cusp form on the upper half plane for $\Gamma_0(N)$, of weight k, and nebentypus character ω. Let $\varphi(z) = \sum_{n=1}^{\infty} a_n q^n$ be the Fourier expansion of φ at infinity. We let $L(s, \varphi)$ stand for the completed L-function associated to φ and let $L_f(s, \varphi)$ stand for its finite part. Assume that φ is a primitive form in $S_k(N, \omega)$. By primitive, we mean that it is an eigenform, a newform and is normalized such that $a_1(\varphi) = 1$. In a suitable right half plane the finite part $L_f(s,\varphi)$ is a Dirichlet series with an Euler product

$$L_f(s,\varphi) = \sum_{n=1}^{\infty} a_n n^{-s} = \prod_p L_p(s,\varphi),$$

where, for all primes p, we have

$$L_p(s,\varphi) = (1 - a_p p^{-s} + \omega(p)p^{k-1-2s})^{-1} = (1-\alpha_{p,\varphi}p^{-s})^{-1}(1-\beta_{p,\varphi}p^{-s})^{-1},$$

with the convention that if $p|N$ then $\beta_{p,\varphi} = 0$. Let η be a Dirichlet character modulo an integer M. We let S_f stand for the set of primes dividing NM and let $S = S_f \cup \{\infty\}$. For any $n \geq 1$, the twisted partial n-th symmetric power L-function of φ is defined as

$$L^S(s, \mathrm{Sym}^n\varphi, \eta) = \prod_{p \notin S} L_p(s, \mathrm{Sym}^n\varphi, \eta),$$

where, for all $p \notin S$, we have

$$L_p(s, \mathrm{Sym}^n\varphi, \eta) = \prod_{i=0}^{n} (1 - \eta(p)\alpha_{p,\varphi}^i \beta_{p,\varphi}^{n-i} p^{-s})^{-1}.$$

If $M = 1$ and η is (necessarily) trivial then we denote the corresponding L-function as $L^S(s, \mathrm{Sym}^n\varphi)$.

Using the local Langlands correspondence the partial L-function can be completed by defining local factors $L_p(s, \mathrm{Sym}^n\varphi, \eta)$ for $p \in S$ and the completed L-function, which is a product over all p including ∞, will be denoted as $L(s, \mathrm{Sym}^n\varphi, \eta)$. The Langlands program predicts that $L(s, \mathrm{Sym}^n\varphi, \eta)$, which is initially defined only in a half plane, admits a meromorphic continuation to the entire complex plane and that it has all the usual properties an automorphic L-function is supposed to have. This is known for $n \leq 4$ from the works of several people including Hecke, Shimura, Gelbart–Jacquet, Kim and Shahidi. It is also known for all n for cusp forms of a special type; for instance, if the representation corresponding to the cusp form is dihedral or the other polyhedral types. (The reader is referred to the same references as in the last paragraph of the previous section.)

3.2 Deligne's conjecture

Let φ be a primitive form in $S_k(N,\omega)$. Let $M(\varphi)$ be the motive associated to φ. This is a rank two motive over $\mathbb{Q}$ with coefficients in the field $\mathbb{Q}(\varphi)$ generated by the Fourier coefficients of φ. (We refer the reader to Deligne [9] and Scholl [39] for details about $M(\varphi)$.) The L-function $L(s, M(\varphi))$ associated to this motive is $L(s,\varphi)$.

Given the motive $M(\varphi)$, and given any complex embedding σ of $\mathbb{Q}(\varphi)$, there are nonzero complex numbers $c^{\pm}(M(\varphi),\sigma)$, called Deligne's periods, associated to it. For simplicity of exposition we concentrate on the case when σ is the identity map, and denote the corresponding periods as $c^{\pm}(M(\varphi))$. We note that this "one component version" of Deligne's conjecture that we state below is not as strong as the full conjecture and so we are dealing with a simplistic situation.

Similarly, for the symmetric powers $\mathrm{Sym}^n(M(\varphi))$, we have the corresponding periods $c^{\pm}(\mathrm{Sym}^n(M(\varphi))$. In [9, Proposition 7.7] the periods for the symmetric powers are related to the periods of $M(\varphi)$. The explicit formulas therein have a quantity $\delta(M(\varphi))$ which is essentially the Gauss sum of the nebentypus character ω and is given by

$$\delta(M(\varphi)) \sim (2\pi i)^{1-k}\mathfrak{g}(\omega) := (2\pi i)^{1-k}\sum_{u=0}^{c-1}\omega_0(u)\exp(-2\pi i u/c),$$

where c is the conductor of ω, ω_0 is the primitive character associated to ω, and by $\sim$ we mean up to an element of $\mathbb{Q}(\varphi)$. We will denote the right-hand side by $\delta(\omega)$. For brevity, we will denote $c^{\pm}(M(\varphi))$ by $c^{\pm}(\varphi)$. Recall [9, Definition 1.3] that an integer m is *critical* for any motivic L-function $L(s,M)$ if both $L_\infty(s,M)$ and $L_\infty(1-s,M^\vee)$ are regular at $s=m$. We now state Deligne's conjecture [9, Section 7] on the special values of the symmetric power L-functions.

Conjecture 3.1. *Let φ be a primitive form in $S_k(N,\omega)$. There exist nonzero complex numbers $c^{\pm}(\varphi)$ such that*

1. *If m is a critical integer for $L_f(s,\mathrm{Sym}^{2l+1}\varphi)$, then*

   $$L_f(m,\mathrm{Sym}^{2l+1}\varphi) \sim (2\pi i)^{m(l+1)}\, c^{\pm}(\varphi)^{(l+1)(l+2)/2}\, c^{\mp}(\varphi)^{l(l+1)/2}\, \delta(\omega)^{l(l+1)/2},$$

   where $\pm = (-1)^m$.
2. *If m is a critical integer for $L_f(s,\mathrm{Sym}^{2l}\varphi)$, then*

   $$L_f(m,\mathrm{Sym}^{2l}\varphi) \sim \begin{cases} (2\pi i)^{m(l+1)}\, (c^+(\varphi)c^-(\varphi))^{l(l+1)/2}\, \delta(\omega)^{l(l+1)/2}, & m \text{ even},\\ (2\pi i)^{ml}\, (c^+(\varphi)c^-(\varphi))^{l(l+1)/2}\, \delta(\omega)^{l(l-1)/2}, & m \text{ odd}.\end{cases}$$

By $\sim$ we mean up to an element of $\mathbb{Q}(\varphi)$.

We wish to emphasize that in the original conjecture of Deligne, the numbers $c^{\pm}$ are *periods*, but in this paper they are just a couple of complex numbers in terms of which of the critical values of the symmetric power L-functions can be expressed.

It adds some clarity to write down explicitly the statement of the conjecture for the n-th symmetric power, in the special cases $n = 1,2,3,4$, and while doing so we also discuss how much is known for these cases.

Let m be a critical integer for $L_f(s,\varphi)$. Then Conjecture 3.1 takes the form

$$L_f(m,\varphi) \sim (2\pi i)^m c^{\pm}(\varphi), \tag{3.1}$$

where $\pm = (-1)^m$. In this context, the conjecture is known and is a theorem of Shimura ([42] and [43]). Shimura relates the required special values to quotients of certain Petersson inner products, whose rationality properties can be studied.

Let m be a critical integer for $L_f(s, \mathrm{Sym}^2\varphi)$. Then Conjecture 3.1 takes the form

$$L_f(m, \mathrm{Sym}^2\varphi) \sim \begin{cases} (2\pi i)^{2m}(c^+(\varphi)c^-(\varphi))\delta(\omega) & \text{if m is even,} \\ (2\pi i)^m(c^+(\varphi)c^-(\varphi)) & \text{if m is odd.} \end{cases} \tag{3.2}$$

The conjecture is known in this case and is due to Sturm ([44] and [45]). Sturm uses an integral representation for the symmetric square L-function due to Shimura [41].

Let m be a critical integer for $L_f(s, \mathrm{Sym}^3\varphi)$. Then Conjecture 3.1 takes the form

$$L_f(m, \mathrm{Sym}^3\varphi) \sim (2\pi i)^{2m}c^{\pm}(\varphi)^3c^{\mp}(\varphi)\delta(\omega), \tag{3.3}$$

where $\pm = (-1)^m$. The conjecture is known in this case and is due to Garrett and Harris [10]. The main thrust of that paper is to prove a theorem on the special values of certain triple product L-functions $L(s, \varphi_1 \times \varphi_2 \times \varphi_3)$. Deligne's conjecture for motivic L-functions predicts the special values of such triple product L-functions, for which an excellent reference is Blasius [2]. Via a standard argument, the case $\varphi_1 = \varphi_2 = \varphi_3 = \varphi$, gives the special values of the symmetric cube L-function for φ. This was reproved by Kim and Shahidi [20] emphasizing finiteness of these L-values which follows from their earlier work [19].

Let m be a critical integer for $L_f(s, \mathrm{Sym}^4\varphi)$. Then Conjecture 3.1 takes the form

$$L_f(m, \mathrm{Sym}^4\varphi) \sim \begin{cases} (2\pi i)^{3m}(c^+(\varphi)c^-(\varphi))^3\delta(\omega)^3 & \text{if m is even,} \\ (2\pi i)^{2m}(c^+(\varphi)c^-(\varphi))^3\delta(\omega) & \text{if m is odd.} \end{cases} \tag{3.4}$$

In general the conjecture is not known for higher ($n \geq 4$) symmetric power L-functions. Although, if φ is dihedral, then we have verified the conjecture for any symmetric power; see Section 4.

We remark that a prelude to this conjecture was certain calculations made by Zagier [47] wherein he showed that such a statement holds for the n-th symmetric power L-function of the Ramanujan Δ-function, with $n \leq 4$.

3.3 Critical points

As recalled above, an integer m is *critical* for any motivic L-function $L(s, M)$ if both $L_\infty(s, M)$ and $L_\infty(1-s, M^\vee)$ are regular at $s = m$. For example, if $M = \mathbb{Z}(0) = H^*(\text{Point})$ is the trivial motive, then $L(s, M)$ is the Riemann zeta function $\zeta(s)$ [9, §3.2]. Then $L_\infty(s, M) = \pi^{-s/2}\Gamma(s/2)$. It is an easy exercise to see that an integer m is critical for $\zeta(s)$ if m is an even positive integer or an odd negative integer. More generally, as in Blasius [2], one can calculate the critical points for any motivic L-function in terms of the Hodge numbers of the corresponding motive. For the specific L-functions at hand, namely the symmetric power L-functions, one explicitly knows the L-factors at infinity [30], using which it is a straightforward exercise to calculate the critical points. In the following two lemmas we record the critical points of the n-th symmetric power L-function associated to a modular form φ. (For more details see [34].)

Lemma 3.2. *Let φ be a primitive cusp form of weight k. The set of critical integers for $L_f(s, \mathrm{Sym}^{2r+1}\varphi)$ is given by integers m with*

$$r(k-1)+1 \leq m \leq (r+1)(k-1).$$

Lemma 3.3. *Let φ be a primitive cusp form of weight k. The set of critical integers for $L_f(s, \mathrm{Sym}^{2r}\varphi)$ is given below.*

1. *If r is odd and k is even, then*
$$\{(r-1)(k-1)+1, (r-1)(k-1)+3, \ldots, r(k-1); \\ r(k-1)+1, r(k-1)+3, \ldots, (r+1)(k-1)\}.$$
2. *If r and k are both odd, then*
$$\{(r-1)(k-1)+1, (r-1)(k-1)+3, \ldots, r(k-1)-1; \\ r(k-1)+2, r(k-1)+4, \ldots, (r+1)(k-1)\}.$$
3. *If r and k are both even, then*
$$\{(r-1)(k-1)+2, (r-1)(k-1)+4, \ldots, r(k-1)-1; \\ r(k-1)+2, r(k-1)+4, \ldots, (r+1)(k-1)-1\}.$$
4. *If r is even and k is odd, then*
$$\{(r-1)(k-1)+1, (r-1)(k-1)+3, \ldots, r(k-1)-1; \\ r(k-1)+2, r(k-1)+4, \ldots, (r+1)(k-1)\}.$$

Remark 3.4. Here are some easy observations based on the above lemmas.

1. If $k=1$ then $L_f(s, \mathrm{Sym}^n\varphi)$ does not have any critical points for any $n \geq 1$. In particular, this is the case if φ is a cusp form which is tetrahedral, octahedral or icosahedral [34].
2. If $k=2$ then $L_f(s, \mathrm{Sym}^n\varphi)$ has a critical point if and only if n is not a multiple of 4; further $L_f(s, \mathrm{Sym}^{2r+1}\varphi)$ has exactly one critical point $m = r+1$; and if r is odd $L_f(s, \mathrm{Sym}^{2r}\varphi)$ has two critical points r and $r+1$. This applies in particular for symmetric power L-functions of elliptic curves.
3. Let m be a critical integer for $L_f(s, \mathrm{Sym}^{2r}\varphi)$. Then m is even if and only if m is to the right of the center of symmetry.

4 Dihedral calculations

A primitive form φ is said to be *dihedral* if the associated cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$, denoted $\pi(\varphi)$, is the automorphic induction of an idèle class character, say χ, of a quadratic extension $K/\mathbb{Q}$. This is denoted as $\pi(\varphi) = \mathrm{AI}_{K/\mathbb{Q}}(\chi)$. (Since φ is a holomorphic modular form, in this situation, K is necessarily an imaginary quadratic extension.) In [34] we give a proof of Deligne's conjecture for the special values of symmetric power L-functions for such dihedral forms using purely the language of L-functions. (See the last paragraph of this section.) In this section we summarize the main results of those calculations while referring the reader to [34] for all the proofs.

Recall from Remark 3.4 that if the weight $k = 1$ then there are no critical integers for $L_f(s, \mathrm{Sym}^n\varphi)$. It is easy to see [34] that if $\pi(\varphi) = \mathrm{AI}_{K/\mathbb{Q}}(\chi)$ and some nonzero power of χ is Galois-invariant (under the Galois group of $K/\mathbb{Q}$) then $k = 1$. Hence we may, and henceforth shall, assume that for every nonzero integer n, χ^n is not Galois-invariant. The following lemma describes the isobaric decomposition of a symmetric power lifting of a dihedral cusp form.

Lemma 4.1. *Let χ be an idèle class character of an imaginary quadratic extension $K/\mathbb{Q}$; assume that χ^n is not Galois-invariant for any nonzero integer n. Let $\chi_{\mathbb{Q}}$ denote the restriction of χ to the idèles of $\mathbb{Q}$. Then we have*

$$\mathrm{Sym}^{2r}(\mathrm{AI}_{K/\mathbb{Q}}(\chi)) = \boxplus_{a=0}^{r-1}\mathrm{AI}_{K/\mathbb{Q}}(\chi^{2r-a}\chi'^{a}) \boxplus \chi_{\mathbb{Q}}^{r},$$
$$\mathrm{Sym}^{2r+1}(\mathrm{AI}_{K/\mathbb{Q}}(\chi)) = \boxplus_{a=0}^{r}\mathrm{AI}_{K/\mathbb{Q}}(\chi^{2r+1-a}\chi'^{a}),$$

where χ' is the nontrivial Galois conjugate of χ.

Note that every isobaric summand above is either cuspidal or is one-dimensional. This lemma can be recast in terms of L-functions. For an idèle class character χ of an imaginary quadratic extension $K/\mathbb{Q}$, we let φ_χ denote the primitive cusp form such that $\pi(\varphi_\chi) = \mathrm{AI}_{K/\mathbb{Q}}(\chi)$. If $\varphi_\chi \in S_k(N, \omega)$ then $\omega\omega_K = \chi_{\mathbb{Q}}$, where we make the obvious identification of classical Dirichlet characters and idèle class characters of $\mathbb{Q}$, and ω_K denotes the quadratic idèle class character of $\mathbb{Q}$ associated to K via global class field theory.

Lemma 4.2. *The symmetric power L-functions of φ_χ decompose as follows:*

$$\begin{aligned} L_f(s, \mathrm{Sym}^{2r}\varphi_\chi) &= L_f(s - r(k-1), (\omega\omega_K)^r) \prod_{a=0}^{r-1} L_f(s - a(k-1), \varphi_{\chi^{2(r-a)}}, \omega^a) \\ &= L_f(s - r(k-1), (\omega\omega_K)^r) \prod_{a=0}^{r-1} L_f(s - a(k-1), \varphi_{\chi^{2(r-a)}}, (\omega\omega_K)^a). \end{aligned}$$

$$\begin{aligned} L_f(s, \mathrm{Sym}^{2r+1}\varphi_\chi) &= \prod_{a=0}^{r} L_f(s - a(k-1), \varphi_{\chi^{2(r-a)+1}}, \omega^a) \\ &= \prod_{a=0}^{r} L_f(s - a(k-1), \varphi_{\chi^{2(r-a)+1}}, (\omega\omega_K)^a). \end{aligned}$$

We can now use the results of Shimura ([42] and [43]) and classical theorems on special values of abelian (degree 1) L-functions for the factors on the right-hand side of the above decompositions to prove Deligne's conjecture on the special values of $L_f(s, \mathrm{Sym}^n \varphi_\chi)$. The proof is an extended exercise in keeping track of various constants after one has related the periods of the cusp form φ_{χ^n} to the periods of φ_χ. We state this as the following theorem.

Theorem 4.3 (Period relations for dihedral forms). *For any positive integer n we have the following relations:*

1. $c^+(\varphi_{\chi^n}) \sim c^+(\varphi_\chi)^n$,
2. $c^-(\varphi_{\chi^n}) \sim c^+(\varphi_\chi)^n \mathfrak{g}(\omega_K)$,

where $\sim$ means up to an element of $\mathbb{Q}(\chi)$—the field generated by the values of χ, and $\mathfrak{g}(\omega_K)$ is the Gauss sum of ω_K.

As mentioned in the introduction, Deligne's conjecture for dihedral forms is known. This is because Deligne's main conjecture is known if one considers only the motives as those attached to abelian varieties and the category as that defined by using absolute Hodge cycles for morphisms. The theorem above can therefore be deduced from the literature. However, the proof, developed by considering the Rankin–Selberg L-function for $\varphi_{\chi^n} \times \varphi_\chi$, is entirely in terms of L-functions. We also use some nonvanishing results for L-functions in the course of proving this theorem.

5 Representations with cohomology

In the study of special values of L-functions, if the L-function at hand is associated to a cuspidal automorphic representation, then a standard assumption made on the representation is that it contributes to cuspidal cohomology. This cohomology space admits a rational structure and certain periods are obtained by comparing this rational structure with a rational structure on the Whittaker model of the representation at hand. This approach to the study of special values is originally due to Harder [12] and since then pursued by several authors and in particular by Mahnkopf [29].

The purpose of this section, after setting up the context, is to record Theorem 5.5 which says that the n-th symmetric power lift of a cohomological cusp form on GL_2, if cuspidal, contributes to cuspidal cohomology of GL_{n+1}. This theorem is essentially due to Labesse and Schwermer [27]. We then digress a little and discuss the issue of functoriality and a representation being cohomological.

5.1 Cohomological representations of $\mathrm{GL}_n(\mathbb{R})$

In this section we set up the context of cohomological representations. This is entirely standard material; we refer the reader to Borel–Wallach [5] and Schwermer [40] for generalities on the cohomology of representations.

We let $G_n = \mathrm{GL}_n$ and B_n be the standard Borel subgroup of upper triangular matrices in G_n. Let T_n be the diagonal torus in G_n and Z_n be the center of G_n. We denote by $X^+(T_n)$ the dominant (with respect to B_n) algebraic characters of T_n. For $\mu \in X^+(T_n)$ let (ρ_μ, M_μ) be the irreducible representation of $G_n(\mathbb{R})$ with highest weight μ. The Lie algebra of $G_n(\mathbb{R})$ will be denoted by $\mathfrak{g}_n$. We let $K_n = \mathrm{O}_n(\mathbb{R})Z_n(\mathbb{R})$ and K_n° be the topological connected component of the identity element in K_n.

Let $\mathrm{Coh}(G_n, \mu)$ be the set of all cuspidal automorphic representations $\pi = \otimes'_{p\leq\infty}\pi_p$ of $\mathrm{GL}_n(\mathbb{A}_\mathbb{Q})$ such that

$$H^*(\mathfrak{g}_n, K_n^\circ\ ;\ \pi_\infty \otimes \rho_\mu) \neq (0).$$

By $H^*(\mathfrak{g}_n, K_n^\circ; -)$ we mean relative Lie algebra cohomology. We recall the following from [5, §I.5.1]: Given a $(\mathfrak{g}_n, K_n)$-module σ, one can talk about $H^*(\mathfrak{g}_n, K_n^\circ; \sigma)$ as well as $H^*(\mathfrak{g}_n, K_n; \sigma)$. Note that $K_n/K_n^\circ \simeq \mathbb{Z}/2\mathbb{Z}$ acts on $H^*(\mathfrak{g}_n, K_n^\circ; \sigma)$ and by taking invariants under this action we get $H^*(\mathfrak{g}_n, K_n; \sigma)$.

Observe that a global representation being cohomological is entirely a function of the representation at infinity. There are two very basic problems, one local and the other global. The latter has given rise to an enormous amount of literature on this theme.

1. The local problem is to classify all irreducible admissible representations π_∞ of $G_n(\mathbb{R})$ which are cohomological, i.e., $H^*(\mathfrak{g}_n, K_n^\circ; \pi_\infty \otimes \rho_\mu) \neq (0)$, and for such representations to actually calculate the cohomology spaces.
2. The global problem is to construct global cuspidal representations whose representation at infinity is cohomological in the above sense.

The reader is referred to Borel–Wallach [5] as a definitive reference for the local problem. For the purposes of this article we discuss the solution of the local problem for tempered representations of $\mathrm{GL}_n(\mathbb{R})$. To begin, we record a very simplified version of [5, Theorem II.5.3] and [5, Theorem II.5.4].

Theorem 5.1. *Let G be a reductive Lie group. Let K be a maximal compact subgroup adjoined with the center of G. Discrete series representations of G (if they exist) are cohomological and have nonvanishing cohomology only in degree* $\dim(G/K)/2$.

We have suppressed any mention of the finite-dimensional coefficients because Wigner's Lemma [5, Theorem I.4.1] gives a necessary condition for the infinitesimal character, and nonvanishing cohomology of a representation pins down the finite-dimensional representation. Here is a well-known example illustrating this theorem.

Example 5.2. Let $G = G_2(\mathbb{R})$ and $K = K_2 = \mathrm{O}_2(\mathbb{R})Z_2(\mathbb{R})$. For any integer $l \geq 1$, we let M_l denote the irreducible representation of G which has dimension l and which is the $(l-1)$-th symmetric power of the standard two-dimensional representation. Let D_l be the discrete series representation of lowest weight $l+1$. (If we take a holomorphic cusp form of weight k then the representation at infinity is D_{k-1}.) The Langlands parameter of D_l is $\mathrm{Ind}_{\mathbb{C}^*}^{W_\mathbb{R}}(\chi_l)$, where $W_\mathbb{R}$ is the Weil group of $\mathbb{R}$, and χ_l is the character of $\mathbb{C}^*$ sending z to $(z/|z|)^l$. The representation D_l is cohomological; more precisely, we have

$$H^q(\mathfrak{g}, K; (D_l \otimes |\cdot|_\mathbb{R}^{-(l-1)/2}) \otimes M_l) = \begin{cases} \mathbb{C} \text{ if } q = 1, \\ 0 \text{ if } q \neq 1. \end{cases}$$

See [46, Proposition I.4(1)] for instance. To compare our notation to the notation therein, take $h = l+1$, $a = l-1$, $\epsilon = 0$ and put $d = (h, a, \epsilon)$. Then our M_l is the $r[d]$ of [46] and our $D_l \otimes |\cdot|_\mathbb{R}^{-(l-1)/2}$ is the $\pi[d]$ of [46]. See [27, §2.1] for an SL_2 version of this example.

It is a standard fact that relative Lie algebra cohomology satisfies a Künneth rule [5, §I.1.3]. Using this fact, one can see that if G is a product of m copies of $\mathrm{GL}_2(\mathbb{R})$ then the representation $D_{l_1} \otimes \cdots \otimes D_{l_m}$ is, up to twisting by a suitable power of $|\cdot|_\mathbb{R}$, cohomological with respect to the finite-dimensional coefficients $M_{l_1} \otimes \cdots \otimes M_{l_m}$.

We now recall, very roughly, a version of Shapiro's lemma for relative Lie algebra cohomology. Consider a parabolically induced representation. The cohomology of the induced representation can be described in terms of the cohomology of the inducing representation. (See [5, Theorem III.3.3, (ii)] for a precise formulation.)

We can now give a reasonably complete picture for tempered representations of $\mathrm{GL}_n(\mathbb{R})$ which are cohomological. See Clozel [8, Lemme 3.14]. We follow the presentation in [29, §3.1].

Let $\mathcal{L}_0^+(G_n)$ stand for the set of all pairs $(w, \mathbf{l})$, with $\mathbf{l} = (l_1, \ldots, l_n) \in \mathbb{Z}^n$ such that $l_1 > \cdots > l_{[n/2]} > 0$ and $l_i = -l_{n-i+1}$, and $w \in \mathbb{Z}$, such that

$$w + \mathbf{l} \equiv \begin{cases} 1 \text{ if } n \text{ is even}, \\ 0 \text{ if } n \text{ is odd}. \end{cases}$$

This set $\mathcal{L}_0^+(G_n)$ will parametrize certain tempered representations defined as follows. For $(w, \mathbf{l}) \in \mathcal{L}_0^+(G_n)$, define the parabolically induced representation $J(w, \mathbf{l})$ by

$$J(w, \mathbf{l}) = \mathrm{Ind}_{P_{2,\ldots,2}}^{G_n}((D_{l_1} \otimes |\cdot|_\mathbb{R}^{w/2}) \otimes \cdots \otimes (D_{l_{n/2}} \otimes |\cdot|_\mathbb{R}^{w/2}))$$

if n is even, and

$$J(w, \mathbf{l}) = \mathrm{Ind}_{P_{2,\ldots,2,1}}^{G_n}((D_{l_1} \otimes |\cdot|_\mathbb{R}^{w/2}) \otimes \cdots \otimes (D_{l_{(n-1)/2}} \otimes |\cdot|_\mathbb{R}^{w/2}) \otimes |\cdot|_\mathbb{R}^{w/2})$$

if n is odd. It is well-known that, up to the twist $|\cdot|_\mathbb{R}^{w/2}$, the representations $J(w, \mathbf{l})$ are irreducible tempered representations of G_n [23, §2].

Now we describe the finite-dimensional coefficients. Let $X_0^+(T_n)$ stand for all dominant integral weights $\mu = (\mu_1, \ldots, \mu_n)$ satisfying the purity condition that there is an integer w, called the weight of μ, such that $\mu_i + \mu_{n-i+1} = w$. The sets $\mathcal{L}_0^+(G_n)$ and $X_0^+(T_n)$ are in bijection via the map $(w, \mathbf{l}) \mapsto \mu = w/2 + \mathbf{l}/2 - \rho_n$ where ρ_n is half the sum of positive roots for GL_n. Let w_n be the Weyl group element of G_n of longest length and let $\mu^\vee = -w_n \cdot \mu$. Then $\rho_{\mu^\vee} \simeq (\rho_\mu)^\vee$ is the contragredient of ρ_μ.

Assume that the pair $(w, \mathbf{l})$ corresponds to μ as above. Using Example 5.2 on the cohomology of discrete series representations, and appealing to the Künneth rule and Shapiro's lemma as recalled above, one can conclude that

$$H^q(\mathfrak{g}_n, K_n^\circ; (J(w, \mathbf{l}) \otimes \mathrm{sgn}^t) \otimes M_{\mu^\vee}) = (0)$$

unless the degree q is in the so-called cuspidal range $b_n \le q \le t_n$. For this case: the bottom degree b_n is given by

$$b_n = \begin{cases} n^2/4 & \text{if } n \text{ is even,} \\ (n^2-1)/4 & \text{if } n \text{ is odd;} \end{cases}$$

the top degree t_n is given by

$$t_n = \begin{cases} ((n+1)^2-1)/4 - 1 & \text{if } n \text{ is even,} \\ (n+1)^2/4 - 1 & \text{if } n \text{ is odd;} \end{cases}$$

and the dimension of $H^q(\mathfrak{g}_n, K_n; (J(w, \mathbf{l}) \otimes \mathrm{sgn}^t) \otimes M_{\mu^\vee})$ is 1 if $q = b_n$ or $q = t_n$. The exponent t of the sign character sgn is in $\{0, 1\}$. If n is even, t plays no role since $J(w, \mathbf{l}) \otimes \mathrm{sgn} = J(w, \mathbf{l})$. If n is odd, t is determined by the weight of μ and the parity of $(n-1)/2$, due to considerations of central character (Wigner's lemma).

To complete the picture one notes that, given M_μ, there is, up to twisting by the sign character, only one irreducible, unitary (up to twisting by $|\cdot|_{\mathbb{R}}^{-w/2}$), generic representation with nonvanishing cohomology with respect to M_μ and this representation is a suitable $J(w, \mathbf{l})$. (See [29, §3.1.3].)

Remark 5.3. Let π be a cohomological cuspidal algebraic ([8, §1.2.3]) automorphic representation of $G_n(\mathbb{A}_\mathbb{Q})$. Then the representation π_∞ at infinity has to be a $J(w, \mathbf{l})$ for some $(w, \mathbf{l}) \in \mathcal{L}_0^+(G_n)$. This can be seen as follows. Since π is cuspidal and algebraic, by the purity lemma [8, Lemme 4.9], we get that the parameter of π_∞ is pure. Since it is cohomological, the finite-dimensional coefficients has a highest weight μ which is also pure, i.e., $\mu \in X_0^+(T_n)$. Further, π_∞ being generic and essentially unitary implies that it is a $J(w, \mathbf{l})$ as above.

Example 5.4. To illuminate this picture we work through the above recipe for the case of a holomorphic cusp form. (We use the notation introduced in Example 5.2 and the previous sections.) Let $\varphi \in S_k(N, \omega)$ and consider the cuspidal automorphic representation $\pi = \pi(\varphi) \otimes |\cdot|^s$. Then $\pi_\infty = \pi(\varphi)_\infty \otimes |\cdot|_{\mathbb{R}}^s = D_{k-1} \otimes |\cdot|_{\mathbb{R}}^s$.

1. If k is even, then the representation π_∞ is a $J(w, \mathbf{l})$ exactly when $w = 2s \in \mathbb{Z}$ and $w + k - 1$ is odd. Hence $s \in \mathbb{Z}$ and $\pi_\infty = J(2s, (k-1, -(k-1)))$. The corresponding dominant weight μ is $(s + (k-2)/2, s - (k-2)/2)$. The representation $M_{\mu^\vee}$ is $M_{k-1} \otimes (\det)^{-s-(k-2)/2}$. (For a dominant weight $\mu = (\mu_1, \mu_2) \in \mathbb{Z}^2$ the rational representation $M_{\mu^\vee}$ is $M_{\mu_1-\mu_2+1} \otimes (\det)^{-\mu_1}$.) Using the fact that $\det = \mathrm{sgn} \otimes |\cdot|_\mathbb{R}$ and that $D_{k-1} \otimes \mathrm{sgn} = D_{k-1}$, we get
$$\begin{aligned}\pi_\infty \otimes M_{\mu^\vee} &= (D_{k-1} \otimes |\cdot|_\mathbb{R}^s) \otimes (M_{k-1} \otimes (\det)^{-s-(k-2)/2}) \\ &= (D_{k-1} \otimes |\cdot|_\mathbb{R}^{-(k-2)/2}) \otimes M_{k-1},\end{aligned}$$
which has nontrivial $(\mathfrak{g}, K)$-cohomology (see Example 5.2).
2. If $k \geq 3$ is odd, then π_∞ is a $J(w, \mathbf{l})$ exactly when $w = 2s$ is an odd integer. Letting $s = 1/2 + r$, with $r \in \mathbb{Z}$, we have $\pi_\infty = J(2r+1, (k-1, -(k-1)))$. The corresponding dominant weight μ is $(r + (k-1)/2, r + 1 - (k-1)/2)$. The representation $M_{\mu^\vee}$ is $M_{k-1} \otimes (\det)^{-r-(k-1)/2}$. In this case we get
$$\begin{aligned}\pi_\infty \otimes M_{\mu^\vee} &= (D_{k-1} \otimes |\cdot|_\mathbb{R}^{1/2+r}) \otimes (M_{k-1} \otimes (\det)^{-r-(k-1)/2}) \\ &= (D_{k-1} \otimes |\cdot|_\mathbb{R}^{-(k-2)/2}) \otimes M_{k-1},\end{aligned}$$
which has nontrivial $(\mathfrak{g}, K)$-cohomology as mentioned before. We have excluded the case $k = 1$, because: firstly, the representation at infinity is not cohomological; and secondly, any symmetric power L-function of a weight one form has no critical points.

We remark that in both cases, the condition π_∞ being a $J(w, \mathbf{l})$ is exactly the condition which ensures that the representation $\pi = \pi(\varphi) \otimes |\cdot|^s$ is regular algebraic in the sense of Clozel [8, §1.2.3 and §3.4].

5.2 Functoriality and cohomological representations

Now we turn to the global problem, namely, to construct a cuspidal automorphic representation whose representation at infinity is cohomological. The specific theorem of interest is the following.

Theorem 5.5. *Let $\varphi \in S_k(N, \omega)$ with $k \geq 2$. Let $n \geq 1$. Assume that $\mathrm{Sym}^n(\pi(\varphi))$ is a cuspidal representation of $\mathrm{GL}_{n+1}(\mathbb{A}_\mathbb{Q})$. Let*
$$\Pi = \mathrm{Sym}^n(\pi(\varphi)) \otimes \xi \otimes |\cdot|^s$$
where ξ is any idèle class character such that $\xi_\infty = \mathrm{sgn}^\epsilon$, with $\epsilon \in \{0, 1\}$, and $|\cdot|$ is the adèlic norm. We suppose that s and ϵ satisfy:

1. *If n is even, then let $s \in \mathbb{Z}$ and $\epsilon \equiv n(k-1)/2 \pmod 2$.*
2. *If n is odd then, we let $s \in \mathbb{Z}$ if k is even, and we let $s \in 1/2 + \mathbb{Z}$ if k is odd. We impose no condition on ϵ.*

Then $\Pi \in \mathrm{Coh}(G_{n+1}, \mu^\vee)$ *where* $\mu \in X_0^+(T_{n+1})$ *is given by*

$$\mu = \left(\frac{n(k-2)}{2} + s, \frac{(n-2)(k-2)}{2} + s, \ldots, \frac{-n(k-2)}{2} + s\right) = (k-2)\rho_{n+1} + s.$$

(Recall that ρ_{n+1} *is half the sum of positive roots of* GL_{n+1}*.) In other words, the representation* $\mathrm{Sym}^n(\pi(\varphi)) \otimes \xi \otimes |\cdot|^s$*, with* ξ *and* s *as above, contributes to the cohomology of the locally symmetric space* $\mathrm{GL}_{n+1}(\mathbb{Q})\backslash \mathrm{GL}_{n+1}(\mathbb{A}_{\mathbb{Q}})/K_f K_{n+1,\infty}^\circ$ *with coefficients in the local system determined by* $\rho_{\mu^\vee}$*, where* K_f *is a deep enough open compact subgroup of* $\mathrm{GL}_{n+1}(\mathbb{A}_{\mathbb{Q},f})$*. (Here* $\mathbb{A}_{\mathbb{Q},f}$ *denotes the finite adèles of* $\mathbb{Q}$*.)*

Proof. See Labesse–Schwermer [27, Proposition 5.4] for an SL_n-version of this theorem. When $k = 2$, the theorem has also been observed by Kazhdan, Mazur and Schmidt [16, p. 99].

We sketch the details for the case when $n = 2r$ is even (the case when n is odd being absolutely similar.) The proof follows by observing that the representation at infinity of $\mathrm{Sym}^n(\pi(\varphi))$ is the representation of $\mathrm{GL}_{n+1}(\mathbb{R})$ whose Langlands parameter is $\mathrm{Sym}^n(\mathrm{Ind}_{\mathbb{C}^*}^{W_\mathbb{R}}(\chi_{k-1}))$ where $\chi_{k-1}(z) = (z/|z|)^{k-1}$. It is a pleasant exercise to calculate a symmetric power of a two-dimensional induced representation; upon completion, one obtains the representation Π_∞ as

$$\begin{aligned}\Pi_\infty &= \mathrm{Ind}_{P_{2,\ldots,2,1}}^{G_{n+1}}(D_{2r(k-1)} \otimes \cdots \otimes D_{2(k-1)} \otimes \mathrm{sgn}^{r(k-1)}) \otimes \xi_\infty \otimes |\cdot|_\mathbb{R}^s \\ &= \mathrm{Ind}_{P_{2,\ldots,2,1}}^{G_{n+1}}(D_{2r(k-1)} \otimes \cdots \otimes D_{2(k-1)} \otimes \mathrm{sgn}^{r(k-1)+\epsilon}) \otimes |\cdot|_\mathbb{R}^s.\end{aligned}$$

We deduce that Π_∞ is a $J(w, \mathbf{l})$ (which, as mentioned before, is equivalent to Π being regular algebraic) exactly when $w = 2s \in \mathbb{Z}$, $r(k-1) + \epsilon$ is even,

$$\mathbf{l} = (2r(k-1), \ldots, 2(k-1), 0, -2(k-1), \ldots, -2r(k-1)) = 2(k-1)\rho_{n+1},$$

and that w is even since $w + \mathbf{l}$ is even. These conditions are satisfied by the hypothesis in the theorem. The weight μ is determined by $\mu = w/2 + \mathbf{l}/2 - \rho_{n+1}$ and the first part of the theorem follows from the discussion in the previous section.

Finally, the relation with the cohomology of locally symmetric spaces follows as in [27, §1] or [29, §3.2]. □

Corollary 5.6. *Let* $\varphi \in S_k(N, \omega)$*. Assume that* $k \geq 2$ *and that* φ *is not dihedral. Then, up to twisiting by a quadratic character,* $\mathrm{Sym}^n(\pi(\varphi))$ *for* $n = 2, 3, 4$*, contributes to cuspidal cohomology.*

Proof. In addition to the above theorem one appeals to the main result of [21]. □

One might view this theorem as an example of the possible dictum that a functorial lift of a cohomological representation is cohomological. This has been used in many instances to construct global representations which contribute to cuspidal cohomology. However, there are several instances where a functorial lift of a cohomological representation is not cohomological. On the other hand, it is an interesting question to ask if the converse of the above dictum is true; namely, if a lift is cohomological, is the pre-image, *a fortiori* cohomological? We end the section by a series of examples illustrating some of these principles. But before doing so, we would like to draw the reader's attention to a conjecture of Clozel [7, §1] which is motivated by the ideas of Labesse and Schwermer [27]. It roughly states that given a tempered cohomological representation at infinity, one can find a global cuspidal automorphic representation whose representation at infinity is the given one.

Example 5.7. The following is a sampling of results–which by no means is to be considered exhaustive–where functoriality is used to produce cohomological representations.

1. Labesse and Schwermer [27] proved the existence of nontrivial cuspidal cohomology classes for SL_2 and SL_3 over any number field E which contains a totally real number field F such that $F = F_0 \subset F_1 \subset \cdots \subset F_n = E$ with each F_{i+1}/F_i either a cyclic extension of prime degree or a non-normal cubic extension. The functorial lifts used were base change for GL_2 and the symmetric square lifting of Gelbart and Jacquet. This was generalized for SL_n over E, in conjunction with Borel [4], with the additional input of base change for GL_n.
2. Motivated by [27], Clozel [7] used automorphic induction and proved the existence of nontrivial cuspidal cohomology classes for SL_{2n} over any number field.
3. Rajan [35], also motivated by [27], proved the existence of nontrivial cuspidal cohomology classes for $\mathrm{SL}_1(D)$ for a quaternion division algebra D over a number field E, with E being an extension of a totally real number field F with solvable Galois closure. Other than base change, he used the Jacquet–Langlands correspondence.
4. Ash and Ginzburg [1, §4] have commented on a couple of examples of cuspidal cohomology classes for GL_4 over $\mathbb{Q}$. The first is by lifting from GSp_4 to GL_4 a weight-3 Seigel modular form. The second is to use automorphic induction from GL_2 over a quadratic extension.
5. Ramakrishnan and Wang [38] used the lifting from $\mathrm{GL}_2 \times \mathrm{GL}_3 \to \mathrm{GL}_6$, due to Kim and Shahidi, to construct cuspidal cohomology classes of GL_6 over $\mathbb{Q}$.

In almost all of the above works, functoriality is used to construct cuspidal representations, and in doing so, one exercises some control over the representations at infinity to arrange for them to be cohomological.

Example 5.8. We construct an example to show that a functorial lift of a cohomological representation need not be cohomological. For an even integer k, take two weight k holomorphic cusp forms φ_1 and φ_2, and let $\pi_i = \pi(\varphi_i)$

for $i = 1, 2$. By Example 5.4 we have that both π_1 and π_2 are cohomological representations. Put $\Pi = \pi_1 \boxtimes \pi_2$ (see Ramakrishnan [36]). Choose the forms φ_1 and φ_2 such that Π is cuspidal; this can be arranged by taking exactly one of them to be dihedral, or by arranging that π_1 is not $\pi_2 \otimes \chi$ for any character χ, by virtue of [37, Theorem 11.1]. It is easy to see that Π_∞ is given by

$$\Pi_\infty = \mathrm{Ind}_{P_{2,1,1}}^{G_4(\mathbb{R})}(D_{2(k-1)} \otimes \mathrm{sgn} \otimes 1\!\!1),$$

where $1\!\!1$ is the trivial representation of $\mathbb{R}^*$. Observe that Π_∞ is not a $J(w, \mathbf{l})$ and hence is not cohomological according to Remark 5.3. (Note that Π, as it stands, is not algebraic, but we can replace Π by $\pi_1 \overset{T}{\boxtimes} \pi_2$ (see [8, Definition 1.10]) and make it algebraic; this replaces Π_∞ by $\Pi_\infty \otimes |\cdot|_{\mathbb{R}}^{1/2}$.) However, note that if we took φ_1 and φ_2 to be in general position (unequal weights) then the lift Π would be cohomological. Similarly, it is possible to construct such an example for the lifting from $\mathrm{GL}_2 \times \mathrm{GL}_3$ to GL_6.

Example 5.9. We would like to mention that in the converse direction the $\mathrm{GL}_2 \times \mathrm{GL}_2$ to GL_4 lifting is well-behaved. Now let φ_i have weight $k_i \geq 1$, for $i = 1, 2$, and assume without loss of generality that $k_1 \geq k_2$. With $\Pi = \pi(\varphi_1) \boxtimes \pi(\varphi_2)$ we have

$$\Pi_\infty = \mathrm{Ind}_{P_{2,2}}^{G_4(\mathbb{R})}(D_{k_1+k_2-2} \otimes D_{k_1-k_2}).$$

Suppose Π_∞ is cohomological, i.e., is a $J(w, \mathbf{l})$, then we would have $k_1 + k_2 - 2 > k_1 - k_2 > 0$, which implies that $k_1 > k_2 \geq 2$, and hence both $\pi(\varphi_1)$ and $\pi(\varphi_2)$ are cohomological.

Example 5.10. It seems to be well-known that given a cuspidal representation π of $\mathrm{GL}_4(\mathbb{A}_\mathbb{Q})$, its exterior square lift $\wedge^2(\pi)$, which by Kim [17] is an automorphic representation of $\mathrm{GL}_6(\mathbb{A}_\mathbb{Q})$, is never cohomological.

6 Special values of L-functions of GL_n: the work of Mahnkopf

6.1 General remarks on functoriality and special values

This section is a summary of some recent results due to Joachim Mahnkopf ([28] and [29]). In this work he proves certain special values theorems for the standard L-functions of cohomological cuspidal automorphic representations of GL_n. In principle one can appeal to functoriality and this work of Mahnkopf to prove new special values theorems. For example, given a cusp form $\varphi \in S_k(N, \omega)$, let $\pi(\varphi)$ denote the cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$. Functoriality predicts the existence of an automorphic representation $\mathrm{Sym}^n(\pi(\varphi))$ of $\mathrm{GL}_{n+1}(\mathbb{A}_\mathbb{Q})$. (See Section 2.) Then it is easy to check that

$$L(s, \mathrm{Sym}^n(\pi(\varphi)) = L(s + n(k-1)/2, \mathrm{Sym}^n \varphi),$$

where the left-hand side is the standard L-function of $\mathrm{Sym}^n(\pi(\varphi))$. Using Mahnkopf's work for the function on the left, one can hope to prove a special values theorem for the function on the right. This is fine in principle, but there are several obstacles to overcome before it can be made to work.

6.2 The main results of Mahnkopf [29]

Let $\mu \in X_0^+(T_n)$ and let $\pi \in \mathrm{Coh}(G_n, \mu)$. We let $L(s, \pi) = \prod_{p \le \infty} L(s, \pi_p)$ be the standard L-function attached to π. Any character χ_∞ of $\mathbb{R}^*$ is of the form $\chi_\infty = \epsilon_\infty |\cdot|^m$ for a complex number m. We say χ_∞ is critical for π_∞ if

1. $m \in 1/2 + \mathbb{Z}$ if n is even, and $m \in \mathbb{Z}$ if n is odd; and
2. $L(\pi_\infty \otimes \chi_\infty, 0)$ and $L(\pi_\infty^\vee \otimes \chi_\infty^{-1}, 1)$ are regular values.

We say $\chi : \mathbb{Q}^* \backslash \mathbb{A}_\mathbb{Q}^\times \to \mathbb{C}^*$ is critical for π if χ_∞ is critical for π_∞. Let $\mathrm{Crit}(\pi)$ stand for all such characters χ which are critical for π. Let $\mathrm{Crit}(\pi)^{\le}$ stand for all $\chi \in \mathrm{Crit}(\pi)$ such that if $\chi_\infty = \epsilon_\infty |\cdot|^m$ then $m \le (1 - \mathrm{wt}(\mu))/2$.

Let $\pi \in \mathrm{Coh}(G_n, \mu)$ and let $\chi \in \mathrm{Crit}(\pi)$. Let $\chi_\infty = \epsilon_\infty |\cdot|^m$. Given $\mu = (\mu_1, \dots, \mu_n) \in X^+(T_n)$ choose a $\lambda = (\lambda_1, \dots, \lambda_{n-1}) \in X^+(T_{n-1})$ such that

1. $\mu_1 \ge \lambda_1 \ge \mu_2 \ge \cdots \ge \lambda_{n-1} \ge \mu_n$; and
2. $\lambda_{n/2} = -m + 1/2$ if n is even, and $\lambda_{(n+1)/2} = -m$ if n is odd.

Proposition 1.1 of [28] says that such a λ exists. Let P be the standard parabolic subgroup of G_{n-1} of type $(n-2, 1)$ and let W^P be a system of representatives for $W_{M_P} \backslash W_{G_{n-1}}$. Let $\widehat{w} \in W^P$ be given by

$$\widehat{w} = \begin{pmatrix} 1 \; 2 \cdots \left[\frac{n}{2}\right] - 1 & \left[\frac{n}{2}\right] & \left[\frac{n}{2}\right] + 1 \cdots n-1 \\ 1 \; 2 \cdots \left[\frac{n}{2}\right] - 1 & n-1 & \left[\frac{n}{2}\right] \quad \cdots n-2 \end{pmatrix}.$$

Define the weight $\mu' = (\widehat{w}(\lambda + \rho_{n-1}) - \rho_{n-1})|_{T_{n-2}} \in X^+(T_{n-2})$ where T_{n-2} is embedded in T_{n-1} as $t \mapsto \mathrm{diag}(t, 1)$.

Theorem 6.1 (Theorem 5.4 in Mahnkopf [29]). *Let $\mu \in X_0^+(T_n)$ be regular and let $\pi \in \mathrm{Coh}(\mathrm{GL}_n, \mu^\vee)$. Let $\mu' \in X^+(T_{n-2})$ be as above and $\pi' \in \mathrm{Coh}(\mathrm{GL}_{n-2}, \mu')$; if n is odd then π' has to satisfy a parity condition. We have*

1. $\mathrm{Crit}(\pi)^{\le} \subset \mathrm{Crit}(\pi')^{\le}$.
2. *Let $\chi \in \mathrm{Crit}(\pi)^{\le}$, with $\chi_\infty = \epsilon_\infty |\cdot|^m$. There exists a collection of complex numbers $\Omega(\pi, \pi', \epsilon_\infty) \in \mathbb{C}^*/\mathbb{Q}(\pi)\mathbb{Q}(\pi')$ such that for any finite extension $E/\mathbb{Q}(\pi)\mathbb{Q}(\pi')$ the tuple $\{\Omega(\pi, \pi', \epsilon_\infty)\}_{\sigma \in \mathrm{Hom}(E, \mathbb{C})} \in (E \otimes \mathbb{C})^*/(\mathbb{Q}(\pi)\mathbb{Q}(\pi'))^*$ is well-defined. There exists a complex number $P_\mu(m)$, depending only on μ and m, subject to Assumption 6.2 below, such that for all $\sigma \in \mathrm{Aut}(\mathbb{C}/\mathbb{Q})$ and almost all χ as above, we have*

$$\left(\frac{\mathfrak{g}(\chi)\mathfrak{G}(\eta) P_\mu(m)}{\Omega(\pi, \pi', \epsilon_\infty)} \frac{L(\pi \otimes \chi\eta, 0)}{L(\pi'^\vee \otimes \chi, 0)} \right)^\sigma = \frac{\mathfrak{g}(\chi^\sigma)\mathfrak{G}(\eta^\sigma) P_\mu(m)}{\Omega(\pi^\sigma, \pi'^\sigma, \epsilon_\infty)} \frac{L(\pi^\sigma \otimes \chi^\sigma \eta^\sigma, 0)}{L((\pi'^\vee)^\sigma \otimes \chi^\sigma, 0)},$$

where η is a certain auxiliary character and $\mathfrak{G}(\eta)$ a certain product of Gauss sums associated to η.

The above theorem is valid only under the following assumption.

Assumption 6.2. $P_\mu(m) \neq 0$.

The quantity $P_\mu(m)$ is the value at $s = 1/2$ of an archimedean Rankin–Selberg integral attached to a certain cohomological choice of Whittaker functions. Mahnkopf proves a necessary condition for this nonvanishing assumption [29, §6]. At present this seems to be a serious limitation of this technique. It is widely believed that this assumption is valid and it has shown up in several other works based on the same, or at any rate similar, techniques. See for instance Ash–Ginzburg [1], Kazhdan–Mazur–Schmidt [16] and Harris [13]. *It is an important technical problem to be able to prove this nonvanishing hypothesis.*

The proof of the above theorem combines both the Langlands–Shahidi and the Rankin–Selberg methods of studying L-functions. One considers the pair of representations $\pi \times \mathrm{Ind}_P^{G_{n-1}}(\pi' \otimes \chi)$ of $G_n(\mathbb{A}_\mathbb{Q}) \times G_{n-1}(\mathbb{A}_\mathbb{Q})$ and carefully chooses a cusp form $\phi \in \pi$ and an Eisenstein series $\mathcal{E}$ corresponding to a section in $\mathrm{Ind}_P^{G_{n-1}}(\pi' \otimes \chi)$. To this pair $(\phi, \mathcal{E})$ a certain Rankin–Selberg type zeta integral [29, 2.1.2], which has a cohomological interpretation, computes the quotient of L-functions appearing in the theorem.

The theorem roughly says that the special values of a standard L-function for GL_n are determined in terms of those of a standard L-function for GL_{n-2}. This descent process terminates since we know the special values of L-functions for GL_1 and GL_2, and we get the following theorem (see [29, §5.5] for making the right choices in the induction on n).

Theorem 6.3 (Theorem A in Mahnkopf [29]). *Assume that $\mu \in X_0^+(T_n)$ is regular and let $\pi \in \mathrm{Coh}(G_n, \mu^\vee)$. Let $\chi \in \mathrm{Crit}(\pi)^{\leq}$. To π and χ_∞ is attached $\Omega(\pi, \chi_\infty) \in \mathbb{C}$ such that for all but finitely many such χ we have*

$$\left(\frac{\mathfrak{g}(\chi)^{[n/2]} \mathfrak{G}(\eta)}{\Omega(\pi, \chi_\infty)} L(\pi \otimes \chi\eta, 0) \right)^\sigma = \frac{\mathfrak{g}(\chi^\sigma)^{[n/2]} \mathfrak{G}(\eta^\sigma)}{\Omega(\pi^\sigma, \chi^\sigma_\infty)} L(\pi^\sigma \otimes \chi^\sigma \eta^\sigma, 0),$$

where η is a certain auxiliary character and $\mathfrak{G}(\eta)$ a certain product of Gauss sums associated to η. Moreover, write $\chi_\infty = \epsilon'_\infty |\cdot|^l_\infty$ and set $\epsilon(\chi_\infty) = \epsilon'_\infty \mathrm{sgn}^l$. There are periods $\Omega_\epsilon(\pi) \in \mathbb{C}^*$ if n is even, and $\Omega(\pi) \in \mathbb{C}^*$ if n is odd, and a collection $P^l_\mu \in \mathbb{C}$, such that $\Omega(\pi, \chi_\infty) = P^l_\mu \Omega(\pi)$ if n is odd, and $\Omega(\pi, \chi_\infty) = P^l_\mu \Omega_{\epsilon(\chi_\infty)}(\pi)$ if n is even.

Note that Theorem 6.3, since it uses Theorem 6.1, also depends on Assumption 6.2.

7 A conjecture on twisted L-functions

The periods c^+ and c^- which appear in Deligne's conjecture are motivically defined. (See Deligne [9, (1.7.2)].) On the other hand, the periods which appear in the work of Harder, and also Mahnkopf, have an entirely different origin, namely, they come by a comparison of rational structures on cuspidal cohomology on the one hand and a Whittaker model for the representation, on the other. See Harder [12, p. 81] and Mahnkopf [29, § 3.4]. It is not at the moment clear how one might explicitly compare these different periods attached to the same object. (See also Remark (2) in Harder's paper [12, p. 85].)

However, one might ask if these different periods behave in the same manner under twisting. Here is a simple example to illustrate this. Let χ be an even Dirichlet character. Let m be an even positive integer. Such an m is critical for $L(s,\chi)$. It is well-known [31, Corollary V.II.2.10] that

$$L_f(m,\chi) \sim_{\mathbb{Q}(\chi)} (2\pi i)^m \mathfrak{g}(\chi).$$

By $\sim_{\mathbb{Q}(\chi)}$ we mean up to an element of the (rationality) field $\mathbb{Q}(\chi)$ generated by the values of χ. Now let η be possibly another even Dirichlet character. Applying the result to the character $\chi\eta$, and using [42, Lemma 8], we get

$$L_f(m,\chi\eta)/L_f(m,\chi) \sim_{\mathbb{Q}(\chi)\mathbb{Q}(\eta)} \mathfrak{g}(\eta).$$

Observe that the *period*, namely the $(2\pi i)^m$, does not show up, and we have the relation that the special value of the twisted L-function and the original L-function differ, up to rational quantities, by the Gauss sum of the twisting character.

Another example along these lines follows from Shimura [43]. Let $\varphi \in S_k(N,\omega)$ and let η be an even Dirichlet character. For any integer m, with $1 \le m \le k-1$, we have

$$L_f(m,\varphi,\eta) \sim_{\mathbb{Q}(\varphi)\mathbb{Q}(\eta)} \mathfrak{g}(\eta) L_f(m,\varphi).$$

The point being that, in the above relation, the periods $c^{\pm}(\varphi)$ do not show up, and so the definition of these periods is immaterial. (One can rewrite this relation entirely in terms of periods of the associated motives and it takes the form $c^{\pm}(M(\varphi)\otimes M(\eta)) \sim \mathfrak{g}(\eta)c^{\pm}(M(\varphi))$, the notation being obvious.)

Even if one cannot prove a precise theorem on special values of L-functions in terms of these–motivically or otherwise defined–periods, one can still hope to prove such period relations. Sometimes such period relations are sufficient for applications; see for instance Murty–Ramakrishnan [25] where such a period relation is used to prove Tate's conjecture in a certain case.

With this motivation, we formulate the following conjecture on the behavior of the special values of symmetric power L-functions under twisting by Dirichlet characters.

Conjecture 7.1. *Let $\varphi \in S_k(N,\omega)$ be a primitive form. Let η be a primitive Dirichlet character.*

1. *Suppose* η *is even, i.e.,* $\eta(-1) = 1$*. Then the critical set for* $L_f(s, \mathrm{Sym}^n\varphi, \eta)$ *is the same as the critical set for* $L_f(s, \mathrm{Sym}^n\varphi)$*, and if* m *is critical, then*

$$L_f(m, \mathrm{Sym}^n\varphi, \eta) \sim \mathfrak{g}(\eta)^{\lceil (n+1)/2 \rceil} L_f(m, \mathrm{Sym}^n\varphi),$$

unless n *is even and* m *is odd (to the left of center of symmetry), in which case we have*

$$L_f(m, \mathrm{Sym}^n\varphi, \eta) \sim \mathfrak{g}(\eta)^{n/2} L_f(m, \mathrm{Sym}^n\varphi).$$

2. *Suppose* η *is odd, i.e.,* $\eta(-1) = -1$*, and* n *is even. Then, if* m *is critical for* $L_f(s, \mathrm{Sym}^n\varphi, \eta)$*, then either* $m+1$ *or* $m-1$ *is critical for* $L_f(s, \mathrm{Sym}^n\varphi)$*. For such an* m *to the right of the center of symmetry we have*

$$L_f(m, \mathrm{Sym}^n\varphi, \eta) \sim ((2\pi i)^{\mp}\mathfrak{g}(\eta))^{n/2+1} L_f(m \pm 1, \mathrm{Sym}^n\varphi),$$

and if m *is to the left of the center of symmetry, we have*

$$L_f(m, \mathrm{Sym}^n\varphi, \eta) \sim ((2\pi i)^{\mp}\mathfrak{g}(\eta))^{n/2} L_f(m \pm 1, \mathrm{Sym}^n\varphi).$$

3. *Suppose* η *is odd, i.e.,* $\eta(-1) = -1$*, and* n *is odd. Then the critical set for* $L_f(s, \mathrm{Sym}^n\varphi, \eta)$ *is the same as the critical set for* $L_f(s, \mathrm{Sym}^n\varphi)$*. Let* $k \geq 3$*. If* m *is critical for* $L_f(s, \mathrm{Sym}^n\varphi, \eta)$*, then either* $m+1$ *or* $m-1$ *is critical for* $L_f(s, \mathrm{Sym}^n\varphi)$*, and for such an* m

$$L_f(m, \mathrm{Sym}^n\varphi, \eta) \sim ((2\pi i)^{\mp}\mathfrak{g}(\eta))^{(n+1)/2} L_f(m \pm 1, \mathrm{Sym}^n\varphi).$$

In all the three cases $\sim$ *means up to an element of* $\mathbb{Q}(\varphi)\mathbb{Q}(\eta)$*.*

Now we elaborate on the heuristics on which we formulated the above conjecture. For $n = 1$ and $n = 2$ this is contained in the theorems of Shimura ([42] and [43]) and Sturm ([44] and [45]) respectively. For $n = 3$, using results on triple product L-functions for which Blasius [2] is a convenient reference and using Garrett–Harris [10, §6], one can verify that the above conjecture is true. Further, for $n \geq 4$ and if φ is dihedral, i.e., $\pi(\varphi) = \mathrm{AI}_{K/\mathbb{Q}}(\chi)$, then the conjecture follows by applying the known cases of $n = 1, 2$ to each summand in the isobaric decomposition in Lemma 4.1. Observe that the exponent $\lceil (n+1)/2 \rceil$ appearing in the conjecture is the number of summands in the isobaric decomposition.

Although we have not checked this, our conjecture should follow from the more general conjectures of Blasius [3, Conjecture L.9.8] and Panchishkin [33, Conjecture 2.3] on the behavior of periods of motives twisted by Artin motives.

It appears that the authors may be able to prove a weaker version of Conjecture 7.1, and so really prove a relation amongst appropriate periods, using Theorem 6.3 of Mahnkopf; at least in the case when $\mathrm{Sym}^n(\pi(\varphi))$ is known to exist as a cuspidal automorphic representation.

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Bounds for Matrix Coefficients and Arithmetic Applications

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Summary. We explain an important result of Hee Oh [20] on bounding matrix coefficients of semi-simple groups and survey some applications.

1 Introduction

Let G be a connected reductive group over a local field F. We denote by $\widehat{\mathsf{G}(F)}$ the unitary dual of $\mathsf{G}(F)$, that is the collection of equivalence classes of irreducible unitary representations of $\mathsf{G}(F)$. $\widehat{\mathsf{G}(F)}$ has a natural topology known as the *Fell Topology* which is described as follows. We will introduce a basis of neighborhoods. Let ρ be any element of $\widehat{\mathsf{G}(F)}$, ϵ a positive number, $\phi_1, \ldots, \phi_n$ the diagonal matrix coefficients of ρ, and C a compact subset of $\mathsf{G}(F)$. We define the open set

$$W(\phi_1, \ldots, \phi_n, C; \epsilon; \rho)$$

to be the set of all $\eta \in \widehat{\mathsf{G}(F)}$ such that there exist $\phi'_1, \ldots, \phi'_n$ each of which is a sum of the diagonal matrix coefficients of η satisfying

$$|\phi'_i(x) - \phi_i(x)| < \epsilon$$

for all $x \in C$ and all $i = 1, \ldots, n$. For more details see [11, 26]. We say $\mathsf{G}(F)$ has *Property T* if the trivial representation is isolated in $\widehat{\mathsf{G}(F)}$. In concrete terms, for a semi-simple group G, this implies that if ρ is a non-trivial irreducible unitary representation of $\mathsf{G}(F)$, and ϕ is a K-finite matrix coefficient of ρ, then ϕ has exponential decay in the sense that will be made explicit in the next section. Here K is a good maximal compact subgroup. Kazhdan [16] has shown that if G is simple and has F-rank of at least two, then it has property T. The purpose of this note is to discuss a uniform quantitative version of Kazhdan's theorem due to Oh [20], and to describe some recent applications to arithmetic problems. Oh's theorem is uniform in several aspects, and this will be essential in applications. We will make this precise in the text. Older results in this direction are contained in [8, 7, 15 and 14]. The monograph [14] contains the treatment of $\mathrm{SL}_n(\mathbb{R})$ using an approach that is similar to the one presented here. For general groups and applications to ergodic theory, see [27].

At least for $\mathrm{GL}(n)$, property T is intimately related to the generalized Ramanujan conjecture. Let me make this a little more explicit. For simplicity suppose we are working with $\mathrm{PGL}(n)$ over a number field F, and suppose T is the split torus consisting of the diagonal matrices. Let $\pi = \otimes_v \pi_v$ be an automorphic representation of $\mathrm{PGL}(n)$ over F. Let v be a place such that π_v is an unramified principal series representation. Suppose π_v is induced from a character χ_v of the torus $\mathsf{T}(F_v)$. Let $\mathcal{C}(\chi_v)$ be the Langlands class of π_v, and suppose ϕ_v is the normalized spherical function associated with χ_v. Let ϖ_v be a local uniformizer for F_v. Then the matrices

$$t_v^k := \begin{pmatrix} I_k & \\ & \varpi_v I_{n-k} \end{pmatrix} \quad k = 1, \ldots, n-1$$

generate $T(F_v)/T(\mathcal{O}_v)$. Then one can show that

$$\phi_v(t_v^k) = \frac{q_v^{\frac{k(n-k)}{2}}}{\mathrm{vol}(\mathsf{K}\mathsf{t}_\mathrm{v}^\mathrm{k}\mathsf{K})} \cdot \mathrm{tr}(\wedge^k \mathcal{C}(\chi_v)^{-1}).$$

Now one can use global results on the description of the automorphic spectrum such as the theorem of Moeglin and Waldspurger, and bounds towards the Ramanujan conjecture due to Luo, Rudnick, and Sarnak to get non-trivial bounds for the spherical function. Note that this is pretty specific to PGL_n. There are of course two parts to this procedure. The first part is expressing the value of the spherical matrix coefficients. This part is perfectly general. One can in fact describe the value of the matrix coefficients explicitly in terms of the fundamental representations of the L-group of the group; this is done in Satake's paper [23]. The next step would be somewhat problematic. For example, even for a small group such as the symplectic group of order four, Howe and Piatetski-Shapiro have constructed automorphic cuspidal representations whose local components are not tempered. For this reason it is not clear how one can get enough cancellation to obtain non-trivial bounds for the value of the spherical matrix coefficients. For a survey of known results about the Ramanujan conjecture, see Sarnak's notes on the Generalized Ramanujan Conjecture. What is surprising in Oh's theorem is that one can in fact get non-trivial bounds for matrix coefficients, spherical or not, without using explicit formulas for matrix coefficients. Furthermore, Oh's theorem applies even when the representation under consideration is not the local component of an automorphic representation, as long as it is infinite-dimensional and the local rank of the group is at least two.

Let us make some remarks on the situation where the rank of the group is one. This is related to the so-called *Property* τ. For groups of rank one, the trivial representation may not be isolated in the unitary dual of the local group; however, not all is lost. To describe what is known in this case, let G be a connected reductive group over a number field F and let $\widehat{\mathsf{G}(F_v)}_{aut}$ be the closure in the Fell topology of $\widehat{\mathsf{G}(F_v)}$ of the set of all π_v which occur as the v-component of some automorphic representation π of G over F. We say the pair (G, v) has *Property* τ if the trivial representation is isolated in $\widehat{\mathsf{G}(F_v)}_{aut}$. Lubotzky and Zimmer have conjectured, and Clozel has proved [4], that if G is semi-simple, then for all v, (G, v) has Property τ. Before saying anything about the proof of Clozel's theorem, let us recall a principle due to Burger, Li, and Sarnak ([2],[3] for the archimedean place and Clozel-Ullmo [6] for the general F_S case). Let H be a semi-simple subgroup of G defined over F. Then for all places v, if $\sigma \in \widehat{\mathsf{H}(F_v)}_{aut}$ then

$$\mathrm{Ind}_{\mathsf{H}(F_v)}^{\mathsf{G}(F_v)}\, \sigma \subset \widehat{\mathsf{G}(F_v)}_{aut}$$

and if $\rho, \pi \in \widehat{\mathsf{G}(F_v)}_{aut}$, then

$$\mathrm{Res}_{\mathsf{H}(F_v)}^{\mathsf{G}(F_v)} \rho \subset \widehat{\mathsf{H}(F_v)}_{aut}$$

and

$$\rho \otimes \pi \subset \widehat{\mathsf{G}(F_v)}_{aut}.$$

Here *Ind* means unitary induction. In these equations, the inclusion should be understood as saying that if a representation is weakly contained in the left-hand side, then it is contained in the right-hand side. Recall that if we say a representation ρ_1 is weakly contained in ρ_2 every diagonal matrix coefficient of ρ_1 can be approximated uniformly on compact sets by convex combinations of diagonal matrix coefficients of ρ_2. Incidently, the proof of this principle as explained in [3] uses the equidistribution of certain Hecke points. Clozel's idea to prove the Property τ is to use the Burger–Li–Sarnak principle in the following fashion. If G is isotropic, then we let H be a root subgroup isomorphic to either SL_2 or PGL_2 for which one can use what's known about the Ramanujan conjecture. If G is anisotropic, then Clozel shows that G contains certain special subgroups for which Property τ can be verified by transferring automorphic representations to the general linear group via the trace formula. Sarnak has conjectured that if $\mathsf{G}(F_v)$ has rank one, then every non-tempered point of $\widehat{\mathsf{G}(F_v)}_{aut}$ is isolated. We refer the reader to Sarnak's survey [21] for various examples and further explanations. In the case of groups of rank one, one can use Clozel's result to obtain explicit bounds for matrix coefficients. For a conjecture and some results that suffice for various applications, see [13].

This paper is organized as follows. In Section 2 we review Oh's important theorem, and give a rough sketch of the proof. In Section 3 we discuss an application to the equidistribution of Hecke points from [5, 12]. The goal of these papers, especially [12], was to apply the equidistribution of Hecke points to problems of distribution of points of interest on algebraic varieties. Here I would like to emphasize the Hecke points themselves, and for that reason I will not discuss these applications. In the last section we will explain a recent theorem on the distribution of rational points on wonderful compactifications of semi-simple groups of adjoint type, verifying a conjecture of Manin. We will sketch two proofs for this theorem, one due to this author joint with Shalika and Tschinkel, and the other due to Gorodnick, Maucourant, and Oh; both argument rely on Oh's theorem. It should be pointed out that in this article we have not stated theorems in their most general forms, and whenever we have proved anything we have restricted our attention to special cases of theorems to avoid technical points. For the most part, our goal has been to introduce ideas. We refer the reader to the references in the bibliography for technical details and general theorems.

I first learned of Oh's work from Sarnak in 2001. This paper, especially the introduction, has been influenced greatly by his ideas particularly those expressed in [21]. Also I must confess that there is nothing new in these notes; everything here has been taken from the sources listed in the bibliography, and my only contribution has been the choice of the material. I wish to thank Hee Oh and the referee for their careful reading of a draft of this paper and for pointing out various inaccuracies and suggesting improvements. Here I thank W.T. Gan, H. Oh, P. Sarnak, J.A. Shalika, and Yu Tschinkel for many useful discussions over the years.

2 Uniform pointwise bounds for matrix coefficients

2.1 A general theorem of Oh

Let k be a non-archimedean local field of $\mathrm{char}(k) \neq 2$, and residual degree q. Let H be the group of k-rational points of a connected reductive split or quasi-split group with $\mathsf{H}/\mathsf{Z}(\mathsf{H})$ almost k-simple. Let S be a maximal k-split torus, B a minimal parabolic subgroup of H containing S, and K a good maximal compact subgroup of H with Cartan decomposition $\mathsf{G} = \mathsf{K}\mathsf{S}(k)^+\mathsf{K}$. Let Φ be the set of non-multipliable roots of the relative root system $\Phi(\mathsf{H}, \mathsf{S})$, and Φ^+ the set of positive roots in Φ. A subset $\mathcal{S}$ of Φ^+ is called a strongly orthogonal system of Φ if any two distinct elements α and α' of $\mathcal{S}$ are strongly orthogonal, that is, neither of $\alpha \pm \alpha'$ belongs to Φ. Define a bi-K-invariant function $\xi_{\mathcal{S}}$ on H as follows: first set

$$n_{\mathcal{S}}(g) = \frac{1}{2} \sum_{\alpha \in \mathcal{S}} \log_q |\alpha(g)|;$$

then

$$\xi_{\mathcal{S}}(g) = q^{-n_{\mathcal{S}}(g)} \prod_{\alpha \in \mathcal{S}} \Big(\frac{(\log_q |\alpha(g)|)(q-1) + (q+1)}{q+1}\Big).$$

The following is a special case of Theorem 1.1 of [20].

Theorem 2.1. *Assume that the semi-simple k-rank of* H *is at least* 2*. Let* $\mathcal{S}$ *be any strongly orthogonal system of* Φ*. Then for any unitary representation* ϱ *of* H *without an invariant vector and with* K*-finite unit vectors* ν *and* ν'*,*

$$|(\varrho(g)\nu, \nu')| \leq (\dim(\mathsf{K}\nu)\dim(\mathsf{K}\nu'))^{\frac{1}{2}} \cdot \xi_{\mathcal{S}}(g),$$

for any $g \in \mathsf{H}$*.*

Here and elsewhere it is not necessary to assume that the groups are quasi-split. One just needs to assume that the given representation does not have an invariant vector under the action of the subgroup H^+ generated by all one-parameter unipotent subgroups. If the group H is simply-connected, then $\mathsf{H} = \mathsf{H}^+$, but in general they may not be the same. The special case considered above is for simplicity. In order to prove the theorem, Oh constructs a subgroup H_α isomorphic to $\mathrm{SL}_2(k)$ or $\mathrm{PGL}_2(k)$ associated to each root α of a strongly orthogonal system, and then shows that every representation ϱ of H restricted to H_α is a direct integral of tempered representations. Note that this gives a bound for the matrix coefficient of $\varrho|_{\mathsf{H}_\alpha}$ in terms of Harish-Chandra functions for SL_2. Then one uses an idea of Howe to glue the information coming from the various H_α. Roughly the idea is this. Suppose we have a group G and a subgroup H which contains the maximal split torus of G. Suppose π is a representation of G. If we know bounds for the K-finite matrix coefficients of π when restricted to H, then since H contains A, we get bounds for the matrix coefficients of π. Oh's insight is that in the setup of the theorem the subgroups H_α provide the framework for applying such an idea. The proposition that makes this possible is the following general fact [20]:

Proposition 2.2. *Let* G *be a connected reductive group over a local field* F*. Let* A, B*, and* K *be respectively a maximal split torus, a minimal parabolic subgroup containing* A*, and a good maximal compact subgroup of* $\mathsf{G}(F)$*. Further, for* $1 \leq i \leq k$*, let* H_i *be a connected reductive subgroup of* G *such that* $\mathsf{H}_i \cap \mathsf{A}$*,* $\mathsf{H}_i \cap \mathsf{B}$*, and* $\mathsf{H}_i(F) \cap \mathsf{K}$ *are respectively a maximal split torus, a minimal parabolic subgroup, and a good maximal compact subgroup of* H_i*. Suppose*

- *for all* $i \neq j$*,* $\mathsf{H}_i \leq C_{\mathsf{G}}(\mathsf{H}_j)$ *and* $\mathsf{H}_i(F) \cap \mathsf{H}_j(F)$ *is a finite subset of* $\mathsf{H}_i(F) \cap \mathsf{K}$*.*
- *for each* i*, there is a bi-*$\mathsf{H}_i(F) \cap \mathsf{K}$*-invariant function* ϕ_i *on* $\mathsf{H}_i(F)$ *such that for each non-trivial irreducible unitary representation* σ *of* $\mathsf{G}(F)$*, the bi-*$\mathsf{H}_i(F) \cap \mathsf{K}$*-finite matrix coefficients of* $\sigma|_{\mathsf{H}_i(F)}$ *are bounded by* ϕ_i*.*

Then for any unitary representation ρ *of* $\mathsf{G}(F)$ *without a non-zero invariant vector under* $\mathsf{G}^+(F)$ *and for any* K*-invariant unit vectors* v, w

$$\left| \langle \rho \left(c \prod_{i=1}^{k} h_i \right) v, w \rangle \right| \leq \prod_{i=1}^{k} \phi_i(h_i)$$

for $h_i \in \mathsf{H}_i(F)$ *and* $c \in \cap_{i=1}^{k} C_{\mathsf{G}(F)}(\mathsf{H}_i(F))$*.*

For applications of a similar strategy to related problems see [17, 18]. A couple of remarks are in order. Notice that Theorem 2.1 is a perfectly local statement and has nothing to do with automorphic representations. The theorem applies to infinite-dimensional irreducible representations. For this reason, when applying the result to local components of automorphic representations in practice, one needs to make sure that for $v \notin S$, the local representations are in fact infinite-dimensional (for example see Proposition 4.4 of [24] where the required dimension result follows from the Strong Approximation). A second remark is that typically one also needs a similar bound on the spherical functions when the semi-simple rank is equal to one. In this case, local considerations do not suffice, as the trivial representation may not be isolated in the unitary dual of the local group. Given a semi-simple group H as above, we know there is a quasi-split group H' which is a global inner form of H. Note that by the standard theorem in Galois cohomology [22], $\mathsf{H}(F_v)$ will be isomorphic to $\mathsf{H}'(F_v)$ for v outside of a finite set of places. In the application discussed in the introduction we need bounds for the matrix coefficients for almost all places. The point is that given a group H the local groups $\mathsf{H}(F_v)$ are rank one at a positive proportion of places only when H is related to either PGL_2 or $\mathsf{U}(2,1)$. For automorphic representations of $\mathsf{U}(2,1)$ we can use the results of Rogawski to transfer the representations to GL(3) where one can apply Oh's theorem. Also see the remarks regarding the Property τ in the introduction.

The application considered in [20] was to calculate the Kazhdan constants of semi-simple groups. In the subsequent sections we will consider applications of more immediate arithmetic interest.

3 Applications to equidistribution of Hecke points

This application is worked out in a wonderful paper of Clozel, Oh, and Ullmo [5], and was later generalized by Gan and Oh [12]. Let us first explain the setup of [5]. Let G be a connected almost simple simply-connected linear algebraic group over $\mathbb{Q}$ with $\mathsf{G}(\mathbb{R})$ non-compact and $\Gamma \subset \mathsf{G}(\mathbb{Q})$ a congruence subgroup. Let $a \in \mathsf{G}(\mathbb{Q})$. For $x \in \Gamma\backslash\mathsf{G}(\mathbb{R})$, we set $T_a x = \{[\Gamma a \Gamma x] \in \Gamma\backslash\mathsf{G}(\mathbb{R})\}$. We also define a *Hecke operator* T_a on $L^2(\Gamma\backslash\mathsf{G}(\mathbb{R}))$ as follows: for any $f \in L^2(\Gamma\backslash\mathsf{G}(\mathbb{R}))$,

$$T_a(f)(x) = \frac{1}{|T_a x|} \sum_{y \in T_a x} f(y).$$

Alternatively $T_a(f)$ can be described by

$$T_a(f)(x) = \frac{1}{\deg a} \sum_{i=1}^{\deg a} f(a_i x)$$

where $a_1, \ldots, a_{\deg a}$ are representatives for the left action of Γ on $\Gamma a \Gamma$. The purpose of [5] is to obtain an estimate for the L^2-norm of the restriction of T_a to the orthogonal complement of constant functions, and to prove the equidistribution of the sets $T_a x$ as $\deg(a) \to \infty$ with rate estimates. For an explanation of the relevance of these results to concrete arithmetic problems, the reader is referred to [6]. For a more thorough exposition of the results of this section, and applications to the distribution of points on spheres, see [19]. For an alternative approach, see [10].

3.1 Adelic Hecke operators

Our first purpose in this section is to give an adelic interpretation of the Hecke operators described above. Let G be a connected almost simple simply-connected linear algebraic group over $\mathbb{Q}$ with $\mathsf{G}(\mathbb{R})$ non-compact and $\Gamma \subset \mathsf{G}(\mathbb{Q})$ a congruence subgroup. The adele group $\mathsf{G}(\mathbb{A})$ will be the restricted topological product of the group $\mathsf{G}(\mathbb{Q}_p)$ with respect to a collection of compact-open subgroups K_p for each finite prime p. If p is an unramified prime for G, we can take K_p to be a hyper-special maximal compact open subgroup of $\mathsf{G}(\mathbb{Q}_p)$. We may assume that for almost all p, $\mathsf{K}_p = \mathsf{G}(\mathbb{Z}_p)$ for a smooth model of G over $\mathbb{Z}[1/N]$ for some integer N. For each finite p, let U_p be a compact-open subgroup of $\mathsf{G}(\mathbb{Q}_p)$. We assume that for almost all p, $\mathsf{U}_p = \mathsf{K}_p$. Set $\mathsf{U}_f = \prod_p \mathsf{U}_p$. A subgroup Γ of the form $\mathsf{G}(\mathbb{Q}) \cap (\mathsf{G}(\mathbb{R}) \times \mathsf{U}_f)$ is called a *congruence subgroup*. It follows from the *strong approximation* that

$$\mathsf{G}(\mathbb{A}) = \mathsf{G}(\mathbb{Q})\mathsf{G}(\mathbb{R})\mathsf{U}_f.$$

As a result the spaces $\mathsf{G}(\mathbb{Q})\backslash\mathsf{G}(\mathbb{A})/\mathsf{U}_f$ and $\Gamma\backslash\mathsf{G}(\mathbb{R})$ are naturally identified, and there is a natural isomorphism

$$\phi : \mathsf{L}^2(\mathsf{G}(\mathbb{Q})\backslash\mathsf{G}(\mathbb{A}))^{\mathsf{U}_f} \to \mathsf{L}^2(\Gamma\backslash\mathsf{G}(\mathbb{R}))$$

given by $\phi(f)(x) = f(x, (e_p)_p)$. Here $\mathsf{L}^2(\mathsf{G}(\mathbb{Q})\backslash\mathsf{G}(\mathbb{A}))^{\mathsf{U}_f}$ is the space of U_f-invariant functions.

For $a \in \mathsf{G}(\mathbb{Q})$, we set $\deg_p(a) = |\mathsf{U}_p\backslash\mathsf{U}_p a \mathsf{U}_p|$. Then it is seen that $\deg a = \prod_p \deg_p a$. We will now define a local Hecke operator $T_{a(p)}$ acting on the space of a right U_f-invariant function on $\mathsf{G}(\mathbb{Q})\backslash\mathsf{G}(\mathbb{A})$. If $\{a_1, \dots, a_n\}$, $n = \deg_p(a)$ is a collection of representatives for $\mathsf{U}_p\backslash\mathsf{U}_p a \mathsf{U}_p$, we set

$$T_{a(p)}(f)((x_q)_q) = \frac{1}{\deg_p(a)} \sum_{i=1}^{\deg_p(a)} f((x_q)_{q\neq p}, (x_p a_i^{-1})).$$

Clearly $T_{a(p)}$ is independent of the choice of the representatives. Furthermore, given $a \in \mathsf{G}(\mathbb{Q})$, for almost all p, $T_{a(p)}$ is the identity operator. Then one can consider the operator $\hat{T}_a = \prod_p T_{a(p)}$ acting on $\mathsf{L}^2(\mathsf{G}(\mathbb{Q})\backslash\mathsf{G}(\mathbb{A}))^{\mathsf{U}_f}$. Then one can show that for any $a \in \mathsf{G}(\mathbb{Q})$, we have

$$\phi(\hat{T}_a(f)) = T_a(\phi(f)).$$

3.2 Equidistribution

As before let G be a connected almost simple simply-connected linear algebraic group over $\mathbb{Q}$ with $\mathsf{G}(\mathbb{R})$ non-compact and $\Gamma \subset \mathsf{G}(\mathbb{Q})$ a congruence subgroup. Our equidistribution statement in this adelic language is the following statement:

Assertion 3.1. *For $f \in \mathsf{L}^2(\mathsf{G}(\mathbb{Q})\backslash\mathsf{G}(\mathbb{A}))^{\mathsf{U}_f}$, we have*

$$\lim_{\deg a \to \infty} \hat{T}_a(f) = \int_{\mathsf{G}(\mathbb{Q})\backslash\mathsf{G}(\mathbb{A})} f\, d\mu.$$

Let $\mathsf{L}^2_0(\mathsf{G}(\mathbb{Q})\backslash\mathsf{G}(\mathbb{A}))$ be the orthogonal complement of the space of constant functions. The Hecke operators preserve this subspace. Let T^0_a be the restriction of $\hat{T}_a$ to L^2_0, and

$$\|T^0_a\| = \sup\{|\langle \hat{T}_a f, h\rangle| \mid f, h \in \mathsf{L}^2_0(\mathsf{G}(\mathbb{Q})\backslash\mathsf{G}(\mathbb{A}))^{\mathsf{U}_f}, \|f\| = \|h\| = 1\}.$$

Then for any $f \in \mathsf{L}^2(\mathsf{G}(\mathbb{Q})\backslash\mathsf{G}(\mathbb{A}))^{\mathsf{U}_f}$, we have

$$\|\hat{T}_a(f) - \int_{\mathsf{G}(\mathbb{Q})\backslash\mathsf{G}(\mathbb{A})} f\, d\mu\| \leq \|T^0_a\| . \|f - \int_{\mathsf{G}(\mathbb{Q})\backslash\mathsf{G}(\mathbb{A})} f\, d\mu\| \leq 2\, \|T^0_a\| . \|f\|.$$

Consequently, in order to prove the assertion, it would suffice to find a bound for $\|T^0_a\|$ that would go to zero as $\deg a$ gets large. This also gives a rate for the convergence. There are also pointwise convergence statements which we will not discuss here.

We now describe the result that gives the connection between bounds for norms of Hecke operators and bounds for matrix coefficients. First a couple of definitions. If ρ_1, ρ_2 are two representations of $\mathsf{G}(\mathbb{Q}_v)$ for some place v, we say ρ_1 is *weakly contained in* ρ_2 if every diagonal matrix coefficient of ρ_1 can be uniformly approximated on compact sets by convex combinations of diagonal matrix coefficients of ρ_2. For each prime p, we let $\widehat{\mathsf{G}}_p$ be the unitary dual of $\mathsf{G}(\mathbb{Q}_p)$, and $\widehat{\mathsf{G}}^{aut}_p \subset \widehat{\mathsf{G}}_p$ the set of unitary representations that are weakly contained in the representations that occur as the p-components of $\mathsf{L}^2(\mathsf{G}(\mathbb{Q})\backslash\mathsf{G}(\mathbb{A})/\mathsf{O}_f)$ for some compact open subgroup $\mathsf{O}_f \subset \mathsf{G}(\mathbb{A}_f)$. Then we have the following elementary but crucial proposition:

Proposition 3.2. *Let G be a connected almost simple simply-connected $\mathbb{Q}$-group with $\mathsf{G}(\mathbb{R})$ non-compact and $\Gamma \subset \mathsf{G}(\mathbb{Q})$ a congruence subgroup of the form $\Gamma = \mathsf{G}(\mathbb{Q}) \cap (\mathsf{G}(\mathbb{R}) \times \prod_p \mathsf{U}_p)$. Suppose that for each finite p, there exists a bi-K_p-invariant positive function F_p on the group $\mathsf{G}(\mathbb{Q}_p)$ such that for any non-trivial $\rho_p \in \mathsf{G}^{aut}_p$ with K_p-finite unit vectors v, w,*

$$|\langle \rho_p(g)v, w\rangle| \leq (\dim\langle \mathsf{K}_p v\rangle \dim\langle \mathsf{K}_p w\rangle)^{1/2} F_p(g)$$

for all $g \in \mathsf{G}(\mathbb{Q}_p)$. Assume moreover that $F_p(e) = 1$ for almost all p. Then for any $a \in \mathsf{G}(\mathbb{Q})$,

$$\|T_a^0\| \leq C \prod_p F_p(a),$$

with $C = \prod_p [\mathsf{K}_p : \mathsf{K}_p \cap \mathsf{U}_p]$.

Before we give a sketch of the proof of the proposition, we note that Oh's result provides a function F_p whenever the $\mathbb{Q}_p$-rank of the group G is larger than or equal to two. For the places where the $\mathbb{Q}_p$-rank is one, we need to use results towards the Ramanujan conjecture as usual.

We now give a sketch of the proof of the proposition. Let S be a large set of places that contains the archimedean place, such that for $v \notin S$, G is unramified over $\mathbb{Q}_v$ and $\mathsf{U}_v = \mathsf{K}_v$. Let $\mathsf{G}_S = \prod_{v \in S} \mathsf{G}(\mathbb{Q}_v)$. As a G_S-module we write

$$\mathsf{L}_0^2(\mathsf{G}(\mathbb{Q})\backslash\mathsf{G}(\mathbb{A})) = \int_X m_x \rho_x \, d\nu(x)$$

where $X = \widehat{\mathsf{G}}_S$, $\rho_x = \prod_{v \in S} \rho_{x(v)}$ is irreducible, m_x is a multiplicity for each $x \in X$, and ν is a measure on X. Further, each $\rho_{x(v)}$ is an irreducible unitary representation of $\mathsf{G}(\mathbb{Q}_v)$. It follows from the strong approximation that for each $v \in S$, $\rho_{x(v)}$ is non-trivial for almost all $x \in X$. Set $\mathcal{L}_0 = \mathsf{L}_0^2(\mathsf{G}(\mathbb{Q})\backslash\mathsf{G}(\mathbb{A}))$ as a $\mathsf{G}(\mathbb{A}_S)$-space. For each $x \in X$, we let $\mathcal{L}_x = \rho_x^{\oplus x}$ be the ρ_x-isotypic component of ρ_x. Then if $v = (v_x)_{x \in X}$ and $w = (w_x)_{x \in X}$, $v_x, w_x \in \mathcal{L}_x$, are elements of $\mathcal{L}_0$ and we have

$$\langle v, w \rangle = \int_X \langle v_x, w_x \rangle \, d\nu(x).$$

The Hecke operator T_a^0 acts on $\mathcal{L}_0^{\mathsf{U}_f}$ by the product $\prod_{p \in S} T_{a(p)}$ where each $T_{a(p)}$ acts as a local Hecke operator on the p-factor $\rho_{x(p)}^{\mathsf{U}_p}$ of ρ_x as follows: if v is U_p-invariant, then

$$T_{a(p)}(v) = \frac{1}{\deg_p(a)} \rho_{x(p)}(\chi_{\mathsf{U}_p a \mathsf{U}_p})(v) = \frac{1}{\deg_p(a)} \int_{\mathsf{G}(\mathbb{Q}_p)} \chi_{\mathsf{U}_p a \mathsf{U}_p}(g) \rho_{x(p)}(g)(v) \, d\mu_p(g)$$

where μ_p is the Haar measure on $\mathsf{G}(\mathbb{Q}_p)$ with $\mu_p(\mathsf{U}_p) = 1$. Consequently, if $\{a_1, \ldots, a_r\}$, $r = \deg_p(a)$ is a collection of representatives for $\mathsf{U}_p \backslash \mathsf{U}_p a \mathsf{U}_p$, then

$$T_{a(p)}(v) = \frac{1}{\deg_p(a)} \sum_{i=1}^r \rho_{x(p)}(a_i) v.$$

Clearly, in order to prove the proposition it would suffice to show that for each finite $p \in S$, and for any U_p-invariant vectors v, w in the space $\rho_{x(p)}$

$$\langle T_{a(p)} v, w \rangle \leq [\mathsf{K}_p : \mathsf{K}_p \cap \mathsf{U}_p] \, F_p(a) \, \|v\| \, . \, \|w\|.$$

It is easy to see that $\langle T_{a(p)} v, w \rangle = \langle \rho_{x(p)} v, w \rangle$; since for almost all x, $\rho_{x(p)}$ is non-trivial, and the dimension of $\mathsf{K}_p v$ and $\mathsf{K}_p w$ are bounded by $[\mathsf{K}_p : \mathsf{K}_p \cap \mathsf{U}_p]$. We thus get the result.

3.3 A generalization

We now explain the results and methods of [12]. The setup is as follows. Let G be a connected reductive linear algebraic group over $\mathbb{Q}$, and let Z be the connected component of the center of G. We assume that $\mathsf{Z}\backslash\mathsf{G}$ is absolutely simple with $\mathbb{Q}$-rank at least one. Let $\overline{G} = \mathsf{Z}(\mathbb{R})^0\backslash\mathsf{G}(\mathbb{R})^0$, and $\mathsf{G}_\mathbb{Q} = \mathsf{G}(\mathbb{Q}) \cap \mathsf{G}(\mathbb{R})^0$. Let $\Gamma \subset \mathsf{G}_\mathbb{Q}$ be an arithmetic subgroup of $\mathsf{G}(\mathbb{R})^0$ such that $\Gamma = \mathsf{G}_\mathbb{Q} \cap \mathsf{U}$ for some compact open subgroup $\mathsf{U} = \prod_p \mathsf{U}_p$ of $\mathsf{G}(\mathbb{A}_f)$. We let $\overline{\Gamma}$ be the image of Γ in $\overline{G}$. $\overline{\Gamma}$ is also a lattice in $\overline{G}$.

Assumption 3.3. We assume that

$$\mathsf{G}(\mathbb{A}) = \mathsf{G}(\mathbb{Q})\mathsf{G}(\mathbb{R})^0\mathsf{U};$$

$$\mathsf{Z}(\mathbb{A}) = \mathsf{Z}(\mathbb{Q})\mathsf{Z}(\mathbb{R})^0(\mathsf{U} \cap \mathsf{Z}(\mathbb{A}_f)).$$

This assumption is satisfied if for example G is simply-connected and Γ is a congruence subgroup, or when G is $\mathbb{Q}$-split and $\Gamma = \mathsf{G}(\mathbb{R})^0 \cap \mathsf{G}(\mathbb{Z})$. Note that in the last example, G is canonically defined over $\mathbb{Z}$, and for that reason $\mathsf{G}(\mathbb{Z})$ makes sense.

Via the diagonal embedding, $\mathsf{G}_\mathbb{Q}$ is viewed as a subgroup of $\mathsf{G}(\mathbb{A}_f)$. For $a \in \mathsf{G}_\mathbb{Q}$, we set

$$\mathsf{G}[a] = \mathsf{G}_\mathbb{Q} \cap \mathsf{U}a\mathsf{U}.$$

There is an obvious map from $\Gamma\backslash\mathsf{G}[a]$ to $\mathsf{U}\backslash\mathsf{U}a\mathsf{U}$, and this maps turns out to be a bijection. If we set

$$\deg(a) = |\Gamma\backslash\mathsf{G}[a]|,$$

$$\deg_p(a) = |\mathsf{U}_p\backslash\mathsf{U}_p a\mathsf{U}_p|,$$

then $\deg(a) = \prod_p \deg_p(a) < \infty$. For any function f on $\overline{\Gamma}\backslash\overline{G}$, we set

$$T_a(f)(g) = \frac{1}{\deg(a)} \sum_{y \in \Gamma\backslash\mathsf{G}[a]} f(yg).$$

$T_a(f)$ is independent of the choice of representatives for $\Gamma\backslash\mathsf{G}[a]$, and is again a function on $\overline{\Gamma}\backslash\overline{G}$. It is again seen that there are local and adelic Hecke operators with compatibility relations as above. As before we have an equidistribution statement as follows: For any $f \in C_c^\infty(\overline{\Gamma}\backslash\overline{G})$ and $x \in \overline{\Gamma}\backslash\overline{G}$, we have

$$\lim_{\deg a \to \infty} T_a(f)(x) = \int_{\overline{\Gamma}\backslash\overline{G}} f(g)\, d\mu_G(g).$$

Here μ_G is the appropriately normalized invariant measure on $\overline{\Gamma}\backslash\overline{G}$. It is also possible to give a rate for this. For simplicity we will describe the rate of convergence in the L^2-sense. Let R_1 (resp. R_2) be the collection of places where the $\mathbb{Q}_p$-rank of $\mathsf{Z}\backslash\mathsf{G}$ is equal to (resp. greater than) one. For each place p, let $\mathcal{S}_p$ be a maximal strongly orthogonal system of positive roots for $\mathsf{G}(\mathbb{Q}_p)$

with respect to some maximal split torus A_p. Define a real valued function ξ on $\mathsf{G}(\mathbb{Q})$ as

$$\xi(g) = \prod_{p \in R_1} \xi_{\mathcal{S}_p}(g)^{\frac{1}{2}} . \prod_{p \in R_2} \xi_{\mathcal{S}_p}(g). \tag{3.1}$$

Then use the first part of Theorem 3.7 of [12] which asserts that there is a constant $C > 0$ such that for any $f \in \mathsf{L}^2(\overline{\Gamma}\backslash\overline{G})$ and $a \in \mathsf{G}(\mathbb{Q})$,

$$\|T_a(f) - \int_{\overline{\Gamma}\backslash\overline{G}} f(g)\, d\mu_G(g)\|_2 \leq C.\|f\|_2.\xi(a).$$

The proof of this theorem follows an argument similar to the theorem of [5] discussed above.

3.4 Homogeneous varieties

This is considered in [12]. Let G and U satisfy the assumptions of 3.3, and let $\mathsf{H} \subset \mathsf{G}$ be a $\mathbb{Q}$-subgroup. Let $\overline{H}$ be the image of $\mathsf{H}(\mathbb{R})^0$ in $\overline{G}$. Assume that $\overline{\Gamma} \cap \overline{H}$ is a lattice in $\overline{H}$. Let $\mu_{\overline{H}}$ be the right $\mathsf{H}(\mathbb{R})^0$-invariant measure on $\overline{H}$ which gives $\overline{\Gamma} \cap \overline{H}\backslash\overline{H}$ volume 1. The measures $\mu_{\overline{G}}$ and $\mu_{\overline{H}}$ induce a unique $\mathsf{G}(\mathbb{R})^0$-invariant measure μ on the homogeneous space $\overline{H}\backslash\overline{G} \cong \mathsf{Z}(\mathbb{R})^0\mathsf{H}(\mathbb{R})^0\backslash\mathsf{G}(\mathbb{R})^0$. Given an integrable function with compact support on $\overline{H}\backslash\overline{G}$, we define a function F on $\overline{\Gamma}\backslash\overline{G}$ by

$$F(g) = \sum_{\gamma \in (\overline{\Gamma} \cap \overline{H})\backslash\overline{\Gamma}} f(\gamma g). \tag{3.2}$$

Clearly, F is integrable and we have

$$\int_{\overline{\Gamma}\backslash\overline{G}} F(g)\, d\mu_G(g) = \int_{\overline{H}\backslash\overline{G}} f(g)\, d\mu(g).$$

It is seen easily that F has compact support if and only if $\overline{\Gamma} \cap \overline{H}$ is cocompact in $\overline{H}$. Here too we have an equidistribution theorem, but only in the weak sense:

Assertion 3.4. *Let f be an integrable function of compact support on $\overline{H}\backslash\overline{G}$, and let F be as above. Then*

1. *For any $\psi \in C_c^\infty(\overline{\Gamma}\backslash\overline{G})$,*

$$\langle T_a F, \psi\rangle \to \langle \mu(f), \psi\rangle \quad \textit{as } \deg a \to \infty.$$

2. *For any $\psi \in C_c^\infty(\overline{\Gamma}\backslash\overline{G})$ and $a \in \mathsf{G}(\mathbb{Q})$,*

$$\langle T_a F - \mu(f), \psi\rangle \leq C_f \,.\, C_\psi \,.\, \xi(a^{-1})^\delta$$

with C_f, C_ψ constants depending on f and ψ respectively. Here $0 < \delta \leq 1$ with equality when $\overline{\Gamma} \cap \overline{H}$ is cocompact in $\overline{H}$.

Note that this is not a direct consequence of the equidistribution statement of Section 3.3 as the function F is not in general smooth of compact support. Statement (1) of the assertion is not hard to prove. The duality properties of Hecke operators imply that

$$\langle T_a F - \mu(f), \psi \rangle = \langle F, T_{a^{-1}} \psi - \mu_{\overline{G}}(\psi) \rangle.$$

The last integral is equal to

$$\int_{\overline{H}\backslash\overline{G}} f(g) \left(\int_{\overline{\Gamma}\cap\overline{H}\backslash\overline{H}} (T_{a^{-1}}(\psi)(hg) - \mu_{\overline{G}}(\psi)) \, d\mu_{\overline{H}}(h) \right) d\mu(g).$$

As $\deg(a^{-1}) = \deg(a)$, we may apply the equidistribution of Hecke operators to ψ and use the dominated convergence theorem to get (1). The pointwise bounds alluded to in Section 3.3 also give (2) in the cocompact case. In the situation where $\overline{\Gamma} \cap \overline{H}$ is not cocompact in $\overline{H}$, the proof of (2) is much more involved and uses reduction theory and Siegel sets.

4 Applications to rational points

This is considered in [13] and [24]. The basic question is to understand the distribution of rational points on Fano varieties. There are a few conjectures concerning these varieties and their rational points; see [1] for a list. The class of varieties considered in the first two papers is the class of *wonderful compactifications* of semi-simple groups of adjoint type ([9]) over number fields. One can in fact verify Manin's conjecture and its generalizations for this class of varieties. For the sake of this exposition we concentrate on a concrete special case of the theorem which is the following. Let G be an arbitrary semi-simple group of adjoint type over a number field F. Let $\varrho : \mathsf{G} \to \mathrm{GL}_N$ be an absolutely irreducible faithful representation of G defined over F. Consider the induced map $\bar{\varrho} : \mathsf{G}(F) \to \mathbb{P}^{N^2-1}(F)$ on rational points. Let H be an arbitrary height function on $\mathbb{P}^{N^2-1}(F)$. Define a counting function H_ϱ on $\mathsf{G}(F)$ by $H \circ \bar{\varrho}$.

Then we are interested in the asymptotic behavior of the following

$$N(T, \varrho, H) = |\{\gamma \in \mathsf{G}(F) \mid H_\varrho(\gamma) \leq T\}| \tag{4.1}$$

as $T \to \infty$. The theorem proved in [13],[24] for this particular case implies that the main term in the formula is of the form $CT^{\mathsf{a}_\varrho}(\log T)^{\mathsf{b}_\varrho - 1}$ with C of arithmetic-geometric nature, and a_ϱ and b_ϱ completely geometric. The approaches of [13] and [24] are different: [13] uses ergodic theoretic methods, while [24] uses height zeta functions and spectral methods. Both of them nonetheless use [20] in a substantial way. Below we sketch the two approaches and try to highlight where exactly Oh's result has been used. While the results of [13, 24] are very general, for this exposition we will explain the arguments in appropriate special cases to avoid technical problems as much as possible.

4.1 Spectral approach of [24]

Here for the sake of exposition we will assume that G is an F-anisotropic inner form of a split semi-simple group of adjoint type defined over F.

We will now sketch the proof. Using Tauberian theorems one deduces the asymptotic properties of $N(T,\varrho,H)$ from the analytic properties of the *height zeta function*

$$\mathcal{Z}_\varrho(s)=\sum_{\gamma\in\mathsf{G}(F)}H_\varrho(\gamma)^{-s}.$$

Actually, we will use the function

$$\mathcal{Z}_\varrho(s,g)=\sum_{\gamma\in\mathsf{G}(F)}H_\varrho(\gamma g)^{-s}.$$

For $\Re(s)\gg 0$, the right-hand side converges (uniformly on compacts) to a function which is holomorphic in s and continuous in g on $\mathbb{C}\times\mathsf{G}(\mathbb{A})$. Since G is F-anisotropic, $\mathsf{G}(F)\backslash\mathsf{G}(\mathbb{A})$ is compact, and if we assume that H_ϱ is right and left invariant under some compact-open subgroup K_0 of $\mathsf{G}(\mathbb{A}_f)$, we get

$$\mathcal{Z}\in\mathsf{L}^2(\mathrm{G}(F)\backslash\mathrm{G}(\mathbb{A}))^{\mathsf{K}_0}.$$

Since G is anisotropic, we have

$$\mathsf{L}^2(\mathsf{G}(F)\backslash\mathsf{G}(\mathbb{A}))=(\widehat{\bigoplus_\pi}\mathcal{H}_\pi)\oplus(\bigoplus_\chi\mathbb{C}_\chi),\tag{4.2}$$

as a Hilbert direct sum of irreducible subspaces. Here the first direct sum is over infinite-dimensional representations of $\mathsf{G}(\mathbb{A})$ and the second direct sum is a sum over all automorphic characters of $\mathsf{G}(\mathbb{A})$. Consequently,

$$\mathsf{L}^2(\mathsf{G}(F)\backslash\mathsf{G}(\mathbb{A}))^{\mathsf{K}_0}=(\widehat{\bigoplus_\pi}\mathcal{H}_\pi^{\mathsf{K}_0})\oplus(\bigoplus_\chi\mathbb{C}_\chi^{\mathsf{K}_0}),\tag{4.3}$$

a sum over representations containing a K_0-fixed vector (in particular, the sum over characters is *finite*). For each infinite-dimensional π occurring in (4.3) we choose an orthonormal basis $\mathcal{B}_\pi=\{\phi_\alpha^\pi\}_\alpha$ for $\mathcal{H}_\pi^{\mathsf{K}_0}$. We have next the following *Automorphic Fourier expansion*

$$\mathcal{Z}_\varrho(s,g)=\sum_\pi\sum_{\phi\in\mathcal{B}_\pi}\langle\mathcal{Z}_\varrho(s,.),\phi\rangle\phi(g)+\sum_\chi\langle\mathcal{Z}_\varrho(s,.),\chi\rangle\chi(g)\tag{4.4}$$

as an identity of L^2-functions. The first step is to show that the series on the right-hand side is normally convergent in g for $\Re s\gg 0$. This follows from the convergence of the spectral zeta function of the Laplace operator. Consequently (4.4) is an identity of continuous functions. Then we can insert $g=e$ to obtain

$$\mathcal{Z}_\varrho(s) = \sum_\pi \sum_{\phi \in \mathcal{B}_\pi} \langle \mathcal{Z}_\varrho(s,.), \phi\rangle \phi(e) + \sum_\chi \langle \mathcal{Z}_\varrho(s,.), \chi\rangle. \tag{4.5}$$

We need to establish a meromorphic continuation of the right-hand side of (4.5) in order to obtain a proof of the main theorem.

The first step is to find a half plane to which the finite sum $\sum_\chi \langle \mathcal{Z}_\varrho(s,.), \chi\rangle$ has an analytic continuation, plus the determination of the right-most pole. This involves a couple of steps. Let $\chi = \otimes_v \chi_v$ be a one-dimensional automorphic representation of G. Let G' be the split group of which G is an inner form. Then by general theory $\mathsf{G}(F_v)$ and $\mathsf{G}'(F_v)$ are isomorphic for almost all v. This gives a local character χ'_v for almost all v. Then one needs to show that there is an automorphic character χ' of G' such that for almost all v the local components of χ' are the χ'_v. This, via the Cartan decomposition, implies a regularization of $\langle \mathcal{Z}_\varrho(s,.), \chi\rangle$ by a product of Hecke L-functions. The Hecke L-functions that appear in this regularization are the compositions $\chi' \circ \check{\alpha}$ for various $\alpha \in \Delta$ (see below for notation).

The second step is to meromorphically continue the inner products $\langle \mathcal{Z}_\varrho, \phi\rangle$ and then show that the sum of the analytically continued functions is holomorphic in an appropriate domain that contains the domain of holomorphy of the sum over characters discussed above. A key ingredient is the computation of the individual inner products $\langle \mathcal{Z}_\varrho, \phi\rangle$. Without loss of generality we can assume that

$$\mathsf{K}_0 = \prod_{v \notin S} \mathsf{K}_v \times \mathsf{K}_0^S,$$

for a finite set of places S. Here for $v \notin S$, K_v is a maximal special open compact subgroup in $\mathsf{G}(F_v)$. After enlarging S to contain all the places where G is not split, we can assume that $\mathsf{K}_v = \mathsf{G}(\mathcal{O}_v)$. In particular, for $v \notin S$ the local representations π_v are spherical. Thus we have a normalized local spherical function φ_v associated to π_v. We have assumed that each ϕ is right K_0-invariant. In conclusion,

$$\langle \mathcal{Z}_\varrho, \phi\rangle = \prod_{v \notin S} \int_{\mathsf{G}(F_v)} \varphi_v(g_v) H_\varrho(g_v)^{-s}\, dg_v \times \int_{\mathsf{G}(\mathbb{A}_S)} H_\varrho(\eta(g_S))^{-s} \int_{\mathsf{K}_0^S} \phi(k\eta(g_S))\, dk\, dg_S.$$

(Here $\eta : \mathsf{G}(\mathbb{A}_S) \to \mathsf{G}(\mathbb{A})$ is the natural inclusion map.) The integral over $\mathsf{G}(\mathbb{A}_S)$ is easy to handle. Our main concern here is the first factor

$$I_\varrho^S(s) = \prod_{v \notin S} \int_{\mathsf{G}(F_v)} \varphi_v(g_v) H_\varrho(g_v)^{-s}\, dg_v. \tag{4.6}$$

Even though there are no non-trivial groups that are both split and anisotropic, we will explain the regularization of this expression in the situation where the group G is split. Let us introduce some notation:

Let G be a split semi-simple group of adjoint type over a number field F. Let T be a split torus in G, and B a Borel subgroup containing T. B then defines an ordering on the set of roots, and this gives a set of simple roots $\Delta = \{\alpha_1, \ldots, \alpha_r\}$. Let 2ρ be the sum of all the positive roots, and define numbers κ_i by $2\rho = \sum_i \kappa_i \alpha_i$. As G is adjoint, there is a collection of one-parameter subgroups $\{\check{\alpha}_1, \ldots, \check{\alpha}_r\}$ such that $(\check{\alpha}_i, \alpha_j) = \delta_{ij}$, with δ_{ij} the Kronecker delta. Let S be a large finite set of places of F containing the places at infinity. In particular we assume that S is large enough so that if $v \notin S$, then $\mathsf{G}(\mathcal{O}_v)$ is a maximal compact subgroup of $\mathsf{G}(F_v)$ and satisfies the Cartan and Iwasawa decompositions. In particular if for each place v, we let $\mathsf{S}(F_v)^+$ be the semi-group generated by $\{\check{\alpha}_1(\varpi_v), \ldots, \check{\alpha}_r(\varpi_v)\}$, then

$$G(F_v) = \mathsf{G}(\mathcal{O}_v)\mathsf{S}(F_v)^+\mathsf{G}(\mathcal{O}_v).$$

This way for each $g \in \mathsf{G}(F_v)$ one gets an r-tuple of non-negative integers $a_v(g) = (a_1(g), \ldots, a_r(g))$. Let $\mathsf{s} = (s_1, \ldots, s_r) \in \mathbb{C}^r$. We set

$$H_v(g, \mathsf{s}) = q_v^{\langle a_v(g), \mathsf{s}\rangle},$$

where $\langle a_v(g), \mathsf{s}\rangle = \sum_i a_i(g)s_i$. Note that for any height function H_ϱ, there is a sequence of integers $(u_1, \ldots, u_r)$, depending on ϱ, such that if we set $\mathsf{s}_\varrho = (u_1 s, \ldots, u_r s)$ then we have

$$H_\varrho(g)^s = H(g, \mathsf{s}_\varrho).$$

Let $\pi = \otimes_v \pi_v$ be an infinite-dimensional irreducible automorphic representation of $\mathsf{G}(\mathbb{A}_F)$ such that for $v \notin S$, π_v has a $\mathsf{G}(\mathcal{O}_v)$ fixed vector. Let φ_v be the normalized spherical function associated to π_v. For $\mathsf{s} \in \mathbb{C}^r$ we set

$$I_v(\mathsf{s}) = \int_{\mathsf{G}(F_v)} \varphi(g) H_v(g, \mathsf{s})^{-1}\, dg,$$

and

$$I_S(\mathsf{s}) = \prod_{v \notin S} I_v(\mathsf{s}).$$

One of the main technical points of [24] is the proof of the existence of a $\mathsf{w} > 0$ such that $I_S(\mathsf{s})$ is holomorphic as a function of several variables on the open set $\mathcal{T}_{-\mathsf{w}} = \{\mathsf{s} = (s_1, \ldots, s_r) \in \mathbb{C}^r \mid \Re s_i > \kappa_i + 1 - \mathsf{w}\}$. One then proves, using the Cartan decomposition and some volume estimates, that it suffices to consider the (simpler) function of several variables

$$I_S^{\mathtt{simple}}(\mathsf{s}) = \prod_{v \notin S} \left(1 + \sum_{i=1}^{r} q_v^{-s_i + \kappa_i} \varphi_v(\check{\alpha}_i(\varpi_v))\right),$$

and prove its holomorphy on $\mathcal{T}_{-\mathsf{w}}$. For details please see [24] or [25] for the exposition of a simple case. It is clear that the result would follow if we knew that there is a universal constant C such that

$$|\varphi_v(\check{\alpha}_i(\varpi_v))| \leq C q_v^{-\mathsf{w}}. \tag{4.7}$$

Case 1: semi-simple rank 1. In this case $\mathsf{G} = \mathrm{PGL}(2)$ or the projective group of a quaternion algebra and any estimate towards the Ramanujan conjecture suffices (and the Jacquet–Langlands correspondence if necessary).

Case 2: semi-simple rank > 1. First we use a strong approximation argument to show that for $v \notin S$, the representation π_v is not one-dimensional, unless π itself is one-dimensional (a similar argument appears in the work of Clozel and Ullmo [6]). Then we apply Oh's result for the cases where the local rank is at least two. This finishes the proof as all but finitely many places have the property that the local group has rank of at least two. In order to treat all possibilities we need to consider the group U(3). Here we use Rogawski's transfer $\mathrm{U}(3) \to \mathrm{GL}_3$. Then we will need the bounds on Langlands classes of cuspidal automorphic representations due to Luo, Rudnick and Sarnak for GL_3.

The meromorphic continuation of the infinite sum over different automorphic representation again follows from analytic properties of the spectral zeta function.

4.2 Ergodic theory approach of [13]

Here, for simplicity, we will assume that G is a connected split simple group of adjoint type of $\mathbb{Q}$-rank larger than two and $F = \mathbb{Q}$. Also assume that G is equipped with an appropriate $\mathbb{Z}$-structure. For simplicity we will further assume that H_ϱ is invariant under $\mathsf{K}_f = \prod_p \mathsf{G}(\mathbb{Z}_p)$. Set

$$B_T := \{g \in \mathsf{G}(\mathbb{A}) \mid H(g) \leq T\}. \tag{4.8}$$

Note that $N(T, \varrho, H) = |B_T \cap \mathsf{G}(\mathbb{Q})|$. Let τ be the Tamagawa measure of the group G, and set $\tau_{\mathsf{G}} = \tau(\mathsf{G}(\mathbb{Q})\backslash\mathsf{G}(\mathbb{A}))$. Then we will sketch the proof of the following theorem:

Theorem 4.1. *We have*

$$|B_T \cap \mathsf{G}(\mathbb{Q})| \sim \frac{1}{\tau_{\mathsf{G}}}.\tau(B_T) \tag{4.9}$$

as $T \to \infty$.

One can in fact get error estimates too, but here we will not worry about that. After this theorem is proved, in order to get an asymptotic formula for $|B_T \cap \mathsf{G}(\mathbb{Q})|$ and consequently for $N(T, \varrho, H)$, one needs to find an asymptotic formula for $\tau(B_T)$. One can use a Tauberian argument, using a theorem of [24], to conclude that

$$\tau(B_T) \sim C T^{\mathsf{a}_\varrho} (\log T)^{\mathsf{b}_\varrho - 1} \tag{4.10}$$

as $T \to \infty$. We will show that Theorem 4.1 follows from the following *mixing theorem*:

Theorem 4.2. *Let* G *be a connected semi-simple split* $\mathbb{Q}$*-group. Then for any* $f_1, f_2 \in \mathsf{L}^2(\mathsf{G}(\mathbb{Q})\backslash\mathsf{G}(\mathbb{A}))^{\mathsf{K}_f}$ *we have*

$$\int f_1(h)f_2(hg)\,d\tau(h) \to \frac{1}{\tau_{\mathsf{G}}}\int f_1\,d\tau\,.\int f_2\,d\tau \tag{4.11}$$

as $g \to \infty$.

In the statement of the theorem, $\mathsf{L}^2(\mathsf{G}(\mathbb{Q})\backslash\mathsf{G}(\mathbb{A}))^{\mathsf{K}_f}$ is the collection of L^2-functions that are right invariant under K_f, and $g \to \infty$ means the following. If G is simple, we say a sequence $\{g_i\}$ of elements of $\mathsf{G}(\mathbb{A})$ is going to infinity, if for every compact set Ω there is an N such that $g_i \notin \Omega$ for $i > N$; for the semi-simple G, the components of g_i in every simple factor should go to infinity. In this theorem too one can get error estimates.

Let us prove Theorem 4.1 assuming Theorem 4.2. We define a function $F_T(g,h)$ on $\mathsf{G}(\mathbb{A}) \times \mathsf{G}(\mathbb{A})$ by

$$F_T(g,h) = \sum_{\gamma\in\mathsf{G}(\mathbb{Q})} \chi_{B_T}(g^{-1}\gamma h). \tag{4.12}$$

Then clearly F_T descends to a function on $\mathsf{G}(\mathbb{Q})\backslash\mathsf{G}(\mathbb{A}) \times \mathsf{G}(\mathbb{Q})\backslash\mathsf{G}(\mathbb{A})$ which we will denote again by F_T. It is easily seen that

$$F_T(e,e) = |B_T \cap \mathsf{G}(\mathbb{Q})|. \tag{4.13}$$

Consequently the proof will be finished if we show

$$F_T(e,e) \sim \frac{1}{\tau_{\mathsf{G}}}\tau(B_T) \tag{4.14}$$

as $T \to \infty$. Theorem 4.2 is used to prove the following lemma:

Lemma 4.3. *For any* $\alpha \in C_c(\mathsf{G}(\mathbb{Q})\backslash\mathsf{G}(\mathbb{A}) \times \mathsf{G}(\mathbb{Q})\backslash\mathsf{G}(\mathbb{A}))^{\mathsf{K}_f\times\mathsf{K}_f}$, *we have*

$$\lim_{T\to\infty}\frac{1}{\tau(B_T)}\int F_T.\alpha\,d(\tau\times\tau) = \frac{1}{\tau_{\mathsf{G}}}\,.\int \alpha\,d(\tau\times\tau). \tag{4.15}$$

It suffices to prove the lemma for α of the form $\alpha_1 \otimes \alpha_2$. To prove the lemma for such functions, we do a straightforward unfolding to obtain

$$\int F_T.\alpha\,d(\tau\times\tau) = \int_{B_T}\langle\alpha_1, g.\alpha_2\rangle\,d\tau(g). \tag{4.16}$$

Since the height function H_ϱ is proper, $g \to \infty$ if an only if $H_\varrho(g) \to \infty$. Hence by the mixing theorem for any $\epsilon > 0$ there exist $T_0 > 0$ such that

$$\left|\langle\alpha_1, g.\alpha_2\rangle - \frac{1}{\tau_{\mathsf{G}}}\,.\int \alpha\,d(\tau\times\tau)\right| < \epsilon \tag{4.17}$$

whenever $H(g) > T_0$. This easily implies the lemma.

To continue, we make the observation that the balls B_T are *asymptotically well-rounded* in the following sense: there exist constants $a_\epsilon \geq 1$ and $b_\epsilon \leq 1$ tending to 1 as $\epsilon \to 0$ such that for all sufficiently small $\epsilon > 0$ we have

$$b_\epsilon \leq \liminf_T \frac{\tau(B_{(1-\epsilon)T})}{\tau(B_T)} \leq \limsup_T \frac{\tau(B_{(1+\epsilon)T})}{\tau(B_T)} \leq a_\epsilon. \tag{4.18}$$

Fix $\epsilon > 0$. Let Ω_ϵ be a symmetric neighborhood of e in $\mathsf{G}(\mathbb{R})$ such that

$$B_T\Omega_\epsilon \subset B_{(1+\epsilon)T} \quad \text{and} \quad B_{(1-\epsilon)T} \subset \cap_{g\in\Omega_\epsilon} B_T.g \tag{4.19}$$

for all T large. Then if we set $\Omega = \Omega_\epsilon \times \mathsf{K}_f$, we have

$$F_{(1-\epsilon)T}(g,h) \leq F_T(e,e) \leq F_{(1+\epsilon)T}(g,h) \tag{4.20}$$

for all $g, h \in \Omega$. Now let $\psi \in C_c(\mathsf{G}(\mathbb{Q})\backslash\mathsf{G}(\mathbb{A}))^{\mathsf{K}_f}$ be a non-negative function with support contained in Ω and such that $\int \psi \, d\tau = 1$. Set $\alpha = \psi \otimes \psi$ as a function on $\mathsf{G}(\mathbb{Q})\backslash\mathsf{G}(\mathbb{A}) \times \mathsf{G}(\mathbb{Q})\backslash\mathsf{G}(\mathbb{A})$. Then (4.20) implies

$$\langle F_{(1-\epsilon)T}, \alpha\rangle \leq F_T(e,e) \leq \langle F_{(1+\epsilon)T}, \alpha\rangle. \tag{4.21}$$

Now Lemma 4.3 combined with (4.18) implies that

$$\frac{b_\epsilon}{\tau_{\mathsf{G}}} \leq \liminf_T \frac{F_T(e,e)}{\tau(B_T)} \leq \limsup_T \frac{F_T(e,e)}{\tau(B_T)} \leq \frac{a_\epsilon}{\tau_{\mathsf{G}}}. \tag{4.22}$$

Letting $\epsilon \to 0$ proves (4.14) and consequently Theorem 4.1.

It remains to prove Theorem 4.2. We define a bi-K-invariant function on $\mathsf{G}(\mathbb{A})$ by

$$\xi((g_p)_p) = \prod_p \xi_{\mathcal{S}_p}(g_p)$$

for $(g_p)_p \in \mathsf{G}(\mathbb{A})$; note that $p = \infty$ is allowed in the above product. Also note that we have picked this ξ as opposed to the one in (3.1) due to our restrictions on the group G. Then $\xi(g) \to 0$ when $g \to \infty$. Then Theorem 4.2 is a consequence of the following theorem which is, in turn, a consequence of Oh's theorem.

Theorem 4.4. *Let G be as above, and let π be an automorphic representation in the orthogonal complement to the one-dimensional representations. Then for any K-finite unit vectors v, w we have*

$$|\langle \pi(g)v, w\rangle| \leq c_0 \,.\, \xi(g) \,.\, (\dim \mathsf{K}v \,.\, \dim \mathsf{K}w)^{\frac{1}{2}}$$

for a constant c_0 which depends only on the group G.

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